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Boundary crossing probabilities for (q, d)-Slepian-processes

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ABSTRACT

For $0 < q < d$ fixed let $W^{[q,d]} = (W_t^{[q,d]})_{t\in[q,d]}$ be a (q, d)-*Slepian-process* defined as centered, stationary Gaussian process with continuous sample paths and covariance

$$C_{W^{[q,d]}}(s, s+t) = \left(1 - \frac{t}{q}\right)^+, \quad q \le s \le s+t \le d.$$

Note that

$$\frac{1}{\sqrt{q}}(B_t - B_{t-q})_{t\in[q,d]},$$

where B_t is standard Brownian motion, is a (q, d)-Slepian-process. In this paper we prove an analytical formula for the boundary crossing probability $\mathbb{P}\left(W_t^{[q,d]} > g(t) \text{ for some } t \in [q, d]\right)$, $q < d \le 2q$, in the case g is a piecewise affine function. This formula can be used as approximation for the boundary crossing probability of an arbitrary boundary by approximating the boundary function by piecewise affine functions.

1. Introduction

The signal plus noise process

$$Y_t = s(t) + B_t, \quad t \in [0, d],$$

where $d > 0$ is a known constant, $s \in L^2[0, d]$ is a deterministic signal and the noise B is standard Brownian motion, is often used in statistics as model for many different situations. By $L^2[0, d]$ we denote the set of square-integrable functions with respect to the Lebesgue-measure on $[0, d]$. In order to monitor the process $(s(t) + B_t)_{t\in[0,d]}$ it is convenient to consider the path of the process during a moving window $[t - q, t]$ of length q moving over time $t \in [q, d]$, where $0 < q < d$ is fixed. Using this information of the windows over time $t \in [q, d]$ a decision on a hypothesis of interest has to be made. Often it is of interest to test the null-hypothesis

$$H_0 : s = c, \quad \text{where } c \in \mathbb{R} \text{ is a known or unknown constant.} \tag{1.1}$$

A simple statistic for each window is the difference $s(t) + B_t - s(t - q) - B_{t-q}$, $t \in [q, d]$, of the observations at the two boundary points of the interval $[t - q, t]$. Under H_0 and by normalizing the variance to 1 we get the stochastic

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process

$$\frac{1}{\sqrt{q}}(B_t - B_{t-q})_{t\in[q,d]}.$$

This process is at least useful to detect increasing or decreasing signals s. Note that the covariance of the above process is given by

$$C(t,t+u) = \left(1-\frac{t}{q}\right)^+, \quad q \le t \le t+u \le d, \tag{1.2}$$

where $a^+ = a$ if $0 \le a$ and $= 0$ if $a < 0$. We call a centered, stationary Gaussian process $W^{[q,d]} = (W_t^{[q,d]})_{t\in[q,d]}$ with continuous paths and covariance function given in (1.2) (q,d)-Slepian process. Slepian (1961) investigated processes of the form $W^{[1,d]}$, $1 < d$. Nowadays these processes are called Slepian processes, see Azaïs and Wschebor (2009). Slepian-processes or modifications of them have been studied by several authors. We refer the interested reader to Mehr and McFadden (1965), Shepp (1966, 1971), Bar-David (1975), Ein-Gal and Bar-David (1975), Cressie (1980), Cressie and Davis (1981), Abrahams (1984, 1986), Chu et al. (1995), Nikitin and Orsingher (2004/2006), Fuchang and Li (2007), Gegg (2013), Liu et al. (2014), Bischoff and Gegg (2016) and the references given therein.

Boundary crossing probabilities are of specific interest for stochastic processes. For instance,

$$\mathbb{P}\left(W_t^{[q,d]} > g(t) \text{ for some } t \in [q,d]\right) \tag{1.3}$$

where $g : [q,d] \to \mathbb{R}$ is a boundary function, is required to establish one-sided tests of Kolmogorov type for the null-hypothesis (1.1).

Note that for $0 < q < d$

$$W^{[q,d]} \overset{\mathcal{D}}{=} \frac{1}{\sqrt{q}}(B_t - B_{t-q})_{t\in[q,d]} \overset{\mathcal{D}}{=} \left(B_{\frac{t}{q}} - B_{\frac{t}{q}-\frac{q}{q}}\right)_{t\in[q,d]}, \tag{1.4}$$

where "$\overset{\mathcal{D}}{=}$" means "identical in law". Hence, by putting $e := \frac{d}{q}$, $u := \frac{t}{q}$ and $h(u) := g(uq)$, $u \in [1,e]$,

$$\mathbb{P}\left(W_t^{[q,d]} > g(t) \text{ for some } t \in [q,d]\right) = \mathbb{P}\left(W_u^{[1,e]} > h(u) \text{ for some } u \in [1,e]\right). \tag{1.5}$$

Therefore the boundary crossing probability for (q,d)-Slepian processes can be restricted to boundary crossing probabilities of Slepian processes $W := W^{[1,e]}$. Nevertheless, we state our results for (q,d)-Slepian processes. For, the $(q,1)$-Slepian processes appear in models with compact experimental region as described at the very beginning of this introduction and $(2,d)$-Slepian bridges are connected with Brownian bridges, see below.

In the literature analytic formulas of boundary crossing probabilities for Slepian processes are only known if the boundary g is a constant or an affine function, see Ein-Gal and Bar-David (1975), Abrahams (1984). It seems to be impossible to show such an analytic result for an arbitrary boundary function g. Therefore, we suggest the following procedure. At first we present an analytic formula for the boundary crossing probability of a (q,d)-Slepian process by considering the boundary function piecewisely. By this formula the boundary crossing probability can be approximated by approximating the boundary function piecewisely by simple functions as, for instance, constant or affine functions. It is important to emphasize that we can show our results for $d \le 2q$ only, because (q,d)-Slepian processes have not the same properties for $d \le 2q$ and $2q < d$, respectively. All proofs are given in the Appendix.

2. Boundary crossing probability

Given an arbitrary boundary function g an approximation of the boundary crossing probability of a $[q,d]$-Slepian-process $W^{[q,d]}$ can be determined by approximating the boundary function g piecewise by simple functions. Wang and Pötzelberger (1997) applied such an approach to Brownian motion. They used affine functions as simple functions. It is more complicated to apply this idea to Slepian processes because Brownian motion is Markovian, Slepian processes are not.

At first we prove an expression for the boundary crossing probability by considering the boundary function piecewisely without simplifying it.

Theorem 2.1. *Let $q,d \in \mathbb{R}$ with $0 < q < d \le 2q$. Let $n+1$, $n \in \mathbb{N}$, fixed points in time $t_0,\ldots,t_n$ with $q = t_0 < t_1 < t_2 < \cdots < t_n = d$, be given and let $c := \frac{q^{(n+1)/2}}{2^n \pi^{(n+1)/2}\sqrt{(3q-d)(d-q)}}$. Then for a measurable function $g : [q,d] \to \mathbb{R}$ holds true*

$$\mathbb{P}\left(W_t^{[q,d]} > g(t) \textit{ for some } t \in [q,d]\right) = 1 - c \cdot \left(\prod_{i=1}^{n-1} \frac{\sqrt{t_{i+1}-q}}{\sqrt{(t_{i+1}-t_i)(t_i-q)}}\right)$$
$$\times \int_{-\infty}^{g(t_0)} \int_{-\infty}^{g(t_n)} \int_{-\infty}^{g(t_{n-1})} \cdots \int_{-\infty}^{g(t_1)} \exp\left[-\frac{q}{4}\left(\frac{(x_0+x_n)^2}{3q-d} + \frac{(x_0-x_n)^2}{d-q}\right)\right]$$