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Hitting probabilities of a class of Gaussian random fields ☆

Wenqing Ni [a,b], Zhenlong Chen [a,*]

[a] School of Statistics and Mathematics, Zhejiang Gongshang University, Hangzhou 310018, China

[b] School of Science, Jimei University, Xiamen 361021, China

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ABSTRACT

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-Gaussian random field whose components are independent copies of a centered Gaussian random field X_0. Under the assumption that the canonical metric $\sqrt{\mathbb{E}(X_0(t) - X_0(s))^2}$ is commensurate with $\gamma(\sum_{j=1}^N \mid t_j - s_j\mid^{H_j})$, where $s = (s_1, \ldots, s_N), t = (t_1, \ldots, t_N) \in \mathbb{R}^N, H_j \in (0, 1), j = 1, 2, \ldots, N$ and $\gamma(r)$ is a non-negative function with some mild conditions, upper and lower bounds on the hitting probabilities of X are obtained. To illustrate our results, several examples of Gaussian random fields are given.

1. Introduction

Many authors have investigated hitting probabilities of stochastic processes and random fields. We refer to Khoshnevisan (2003) for an excellent survey on Brownian motion; to Khoshnevisan and Shi (1999) for the (N, d)-Brownian sheet; to Xiao (1999) for fractional Brownian motion; to Chen (2013) for the multiparameter multifractional Brownian motion; to Biermé et al. (2009) and Xiao (2009) for a class of anisotropic Gaussian random fields; to Dalang et al. (2007, 2009, 2013) for solutions of systems of nonlinear stochastic heat equations with different kinds of conditions. Recently, Chen and Xiao (2012) have extended and refined the results in Biermé et al. (2009) and Xiao (2009).

Recently Nualart and Viens (2014) have constructed a Gaussian process $X_0 = \{X_0(t), t \in \mathbb{R}\}$ such that the canonical metric $\sqrt{\mathbb{E}(X_0(t) - X_0(s))^2}$ is commensurate with $\gamma(|t - s|)$, where $\gamma(r)$ is continuous and concave, with $\gamma(0) = 0$ and $\gamma'(0+) = +\infty$, and studied its hitting probabilities. In this paper, we consider the hitting probabilities of N-parameter Gaussian random fields whose variograms are commensurate with $\gamma(\sum_{j=1}^N |t_j - s_j|^{H_j})$. Our results allow the time variables to be restricted to Borel sets and also extend those of Biermé et al. (2009) by allowing more flexible covariance structures.

The rest of this paper is organized as follows. In Section 2, we describe the class of Gaussian random fields that we will study and prove several lemmas that will be used in the sequel. In Section 3, we provide upper and lower bounds for the hitting probabilities of the Gaussian random field. To illustrate our main results, we provide several interesting examples in Section 4.

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* Corresponding author.
E-mail address: zlchen@zjgsu.edu.cn (Z. Chen).

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We end the Introduction with some notations. Throughout this paper, the inner product is denoted as $\langle \cdot, \cdot \rangle$, and $|\cdot|$ is used to denote the Euclidean norm in $\mathbb{R}^n$, no matter what the value of n is. For two functions f and g, the notation $f(t) \asymp g(t)$ for $t \in T$ means that the function $f(t)/g(t)$ is bounded from below and above by positive constants that do not depend on $t \in T$.

We will use c to denote an unspecifical positive and finite constant which may not be the same in each occurrence. More specific constants in section i are numbered as $c_{i,1}, c_{i,2}, \ldots$.

2. Preliminaries

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field in $\mathbb{R}^d$ defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ by

$$X(t) = (X_1(t), \ldots, X_d(t)), \quad t \in \mathbb{R}^N, \tag{2.1}$$

where $X_1, \ldots, X_d$ are independent copies of X_0. We assume that X_0 is a continuous mean zero Gaussian random field with $X_0(0) = 0$ a.s. and satisfies the following conditions: There exists a constant $l > 1$, such that

$$\frac{1}{l}\rho_\gamma^2(s,t) \leqslant \mathbb{E}|X_0(t) - X_0(s)|^2 \leqslant l\rho_\gamma^2(s,t), \tag{2.2}$$

where

$$\rho_\gamma(s,t) = \gamma\Big(\sum_{j=1}^N |t_j - s_j|^{H_j}\Big), \quad \forall s, t \in \mathbb{R}^N \tag{2.3}$$

with $H = (H_1, H_2, \ldots, H_N) \in (0,1)^N$, and $\gamma(r)$ is a continuous, strictly increasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ with following conditions:

(C1) There exist constants $m, r_0 > 0$, such that for all $r \in [0, r_0]$, $\int_0^{\frac{1}{2}} \gamma(rv)\frac{1}{v\sqrt{\log\frac{1}{v}}}dv \leqslant m\gamma(r)$.
(C2) $\lim_{r\to 0^+} \gamma(r)\sqrt{\log\frac{1}{r}} = 0$.
(C3) $\gamma(r)$ is concave and has continuous derivative.

Remark 2.1. The following are some remarks about the above conditions.

(1) In Condition (C3), the condition that $\gamma(r)$ is concave is used to assure that ρ_γ is a metric. However, we only need $\gamma(r)$ to be concave in a neighborhood near the origin, since only the small coverings are taken into account for Hausdorff measure and the points with small distance are taken into consideration for capacity in this paper.
(2) From Condition (C3), we have $\lim_{r\to 0^+} \gamma(r) = 0$, Hence, we will assume $\gamma(0) = 0$ for convenience.

For simplicity, we will denote $\mathrm{Var}(X_0(t))$ by $\sigma^2(t)$ and assume $\sigma(t)$ has continuous first order partial derivatives on I throughout this paper.

Next we provide some lemmas which will be used in the sequel. For simplicity of notation, we always assume

$$I = [a, b], \quad \text{for all } 0 < a_j < b_j < \infty,\ j = 1, \ldots, N. \tag{2.4}$$

We also suppose $0 < H_1 \leqslant H_2 \leqslant \cdots \leqslant H_N < 1$, set $Q = \sum_{j=1}^N \frac{1}{H_j}$ and denote the diameter of the rectangle I in metric ρ_γ by h, that is,

$$h = \sup\big\{\rho_\gamma(s,t) : (s,t) \in I\big\} \tag{2.5}$$

in the rest of this paper.

Lemma 2.2 is known as two points local nondeterminism of Gaussian random fields.

Lemma 2.2. *For every I defined in (2.4), there exist $\delta > 0$ and a positive constant $c_{2,1}$ depending only on I and $H_1, H_2, \ldots, H_N$ such that for all $s, t \in I$ with $|t - s| < \delta$,*

$$Var(X_0(t)|X_0(s)) \geqslant c_{2,1}\rho_\gamma^2(s,t). \tag{2.6}$$

Proof. Note that

$$\begin{aligned} \mathrm{Var}(X_0(t)|X_0(s)) &= \mathbb{E}(X_0(t))^2\Bigg(1 - \Big(\frac{\mathbb{E}(X_0(s)X_0(t))}{\sqrt{\mathbb{E}(X_0(s))^2\mathbb{E}(X_0(t))^2}}\Big)^2\Bigg) \\ &= \sigma^2(t)\Big(1 - \frac{\mathbb{E}(X_0(s)X_0(t))}{\sigma(s)\sigma(t)}\Big)\Big(1 + \frac{\mathbb{E}(X_0(s)X_0(t))}{\sigma(s)\sigma(t)}\Big). \end{aligned} \tag{2.7}$$