Statistics and Probability Letters 118 (2016) 171–176

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

# Recovering a distribution from its translated fractional moments

H. Gzyl [a], A. Tagliani [b,*]

[a] Centro de Finanzas IESA, Caracas, Venezuela
[b] Department of Economics and Management, University of Trento, 38100 Trento, Italy

ARTICLE INFO

*Article history:*
Received 17 March 2015
Accepted 23 June 2016
Available online 6 July 2016

*Keywords:*
Entropy
Entropy convergence
Fractional moment
Kullback–Leibler distance
Maximum entropy

ABSTRACT

We take up the problem of determining the distribution of the hitting time of a parabola by a Brownian motion. We use the maximum entropy method to obtain a good approximation to the true density from its translated fractional moments.


## 1. Introduction

In Shepp (1967), the author considered the problem of determining the distribution of the hitting time of a parabola by a Brownian motion issued from the origin. To establish notations and to describe the problem, let $\{B(t)|t \geq 0\}$ be a standard Brownian motion on the real line, and consider the parabola $x^2 - c^2(a+t) = 0$ in $[0, \infty) \times \mathbb{R}$. Denote by $g(t) = \pm c(a+t)^{1/2}$ the two branches of the parabola. Let $T := \inf\{t > 0 | B(t) = g(t)\}$ be the first time that the Brownian motion touches the parabola. In that paper, Shepp proved that the optional sampling theorem can be invoked as we do to conclude that $\mathbb{E}[\exp(-\lambda^2 T/2) \exp(\lambda B(T))] = 1$. This can be rewritten as

$$\int_0^\infty p_a(t) \exp(-\lambda^2 t/2) \cosh c\lambda(a+t)^{1/2} dt = 1$$

in which we use $p_a(t)$ for the density of $T$ to indicate the dependence on $a$, and when $a = 1$ we shall use $p(t)$ instead of $p_1(t)$. Shepp proceeds by multiplying both sides of that expression by $\lambda^\beta \exp(-\lambda^2 a/2)$ and integrating with respect to $\lambda$ over $[0, \infty)$ to obtain

$$\int_0^\infty p_a(t)(a+t)^\mu dt = \int_0^\infty \lambda^\beta \exp(-\lambda^2 a/2) d\lambda \Big/ \int_0^\infty \lambda^\beta \exp(-\lambda^2/2) \cosh c\lambda d\lambda. \tag{1.1}$$

There $\mu = -(1+\beta)/2$ and $\beta > -1$. The expression can be extended to $\mu > 0$ (or $\beta < -1$) by analytic continuation, and it is rewritten as

$$\left(\int_0^\infty p_a(t)(a+t)^\mu dt\right)^{-1} = a^{-\mu} \sum_{m=0}^{\infty} (-2c^2)^m \mu(\mu-1)\ldots(\mu-m+1)/(2m)! \tag{1.2}$$

* Corresponding author.
*E-mail addresses:* henryk.gzyl@iesa.edu.ve (H. Gzyl), aldo.tagliani@unitn.it (A. Tagliani).

http://dx.doi.org/10.1016/j.spl.2016.06.024
0167-7152/

The series in the right hand side of (1.2), is that of the confluent hypergeometric-function (Abramovitz and Stegun, 1965), denoted by $a^{-\mu}M(-\mu, \frac{1}{2}, \frac{c^2}{2})$. Observe that the inverse of the left hand side of (1.2) is the translated fractional moments of $p_a$, which happen to be finite whenever $c < c_0 = c_0(\mu)$, independently of the value of $a$, where $c_0$ is the first (positive) zero of $M(-\mu, \frac{1}{2}, \frac{c^2}{2})$. Also, the $n$th moment of $T$ is finite whenever $c$ is less that the first (positive) zero of the Hermite polynomial $He_{2n}$. See Abramovitz and Stegun (1965) for the tabulation of these zeros.

Besides Sheep's work on the subject, see also Breiman (1996) and Darling and Siegert (1953) for different approaches to the problem. For related problems, applications and more recent work, see Durbin (1988), Durbin (1992), Bañuelos et al. (2001), Peskir (2002), Novikov et al. (2003), Borovkov and Novikov (2005), Cheng et al. (2006), Wang and Pötzelberger (2007), Masaku (2008), Kahale (2008), Keener (2013) and Aksop et al. (2013) for a review of results on related problems. Our aim here will be to determine an approximation $p_N(t)$ to $p(t)$ from the values of $M(-\mu, \frac{1}{2}, \frac{c^2}{2})$ for negative, fractional (i.e., non-integral) values of $\mu$. For that we shall use the method of maximum entropy (MaxEnt for short).

Once the approximation $p_N(t)$ has been obtained, we shall compare the result of computing the true moment curve $\mu \to \int_0^\infty (1+t)^\mu p_a(t)dt = a^\mu / M(-\mu, \frac{1}{2}, \frac{c^2}{2})$ with the approximate one $\mu \to \int_0^\infty (1+t)^\mu p_N(t)dt$.

The remainder of the paper is organized as follows: before closing this section, we prove in two different ways that $p_a(t) = a^{-1}p_1(t/a)$. In Section 2 we argue that a sequence of shifted fractional moments determines $p(t)$ uniquely. Therefore, through a previously formulated criterion choice, it makes sense to consider a few fractional moments (the ones minimizing the entropy) to approximate $p(t)$ numerically. In Section 3 we review the basics of the maximum entropy method as well as the procedure to determine which moments to use, and in Section 4 we present the numerical results. There we shall examine two quantitative criteria of performance of the method. On one hand we measure the quality of the reconstruction for each number of moments by the difference between the true and reconstructed moment curves, and on the other hand we use the entropy convergence results mentioned in Section 3 as criteria for the quality of the approximation of the reconstructed density to the true density. We end with some concluding remarks.

### 1.1. The dependence of $p_a(t)$ on $a$

There are two ways of realizing that $p_a(t) = a^{-1}p_1(t/a)$. Let us say a few words about them. To begin with, note that

$$T_a = \inf\{T > 0 | B(t) = \pm(a+t)^{1/2}\} = a \inf\{t > 0 | a^{-1/2}B(ta) = \pm(1+t)^{1/2}\}.$$

As $\hat{B}(t) := a^{-1/2}B(ta)$ equals $B(t)$ in distribution, it is clear that $T_a/a \sim T_1$ in distribution, therefore

$$F_a(t) = P(T_a \le t) = P(T_1 \le t/a) = F_1(t/a) \Rightarrow p_a(t) = ap_1(t/a).$$

If we use the short hand $\mathbb{E}[(a+T_a)^\mu] = a^\mu/M$ for the right hand of (1.2), clearly

$$\mathbb{E}[(1+T_a/a)^\mu] = \mathbb{E}[(1+T_1)^\mu]/M$$

which combined with a simple change of variables, yields the claim. All of this is to justify choosing $a = 1$ in the section about numerical results and forgetting about the matter. Recall that we shall use $p(t)$ instead of $p_1(t)$.

**Comment** The crossing time of $\pm c(1+t)^{1/2}$ by the standard Brownian motion is the same as the crossing time of $2\int_0^t B(s)dB(s) - (c^2-1)t$ of the level $c^2$. This observation does not seem to lead to a simplified computation of the distribution of $T_1$.

## 2. Lin's determination results

Our setup is a trivial minor variation on the setup considered by Lin (1992, Theorem 2). There he proved that,

**Theorem 2.1.** *If $Z$ is a r.v. assuming values from a bounded interval $[0, 1]$ and $\{\alpha_n\}_{n=0}^\infty$ an infinite sequence of positive and distinct numbers satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^\infty \alpha_n = +\infty$ then the sequence of moments $\{\mathbb{E}(Z^{\alpha_n})\}_{n=0}^\infty$ characterizes $Z$.*

The result hinges on the fact that an analytical function is determined by its values at a sequence of points having an accumulation point in its domain of analyticity.

To make use of Lin's results we introduce the change of variables $Z = \frac{1}{1+T}$, having support $[0, 1]$ and density $f_Z(z)$, related to $p(t)$ by $p(t) = \frac{1}{(1+t)^2} f_Z(\frac{1}{1+t})$. Let $\alpha > 0$ and $\mu = -\alpha$, so that Lin theorem may be used because

$$\mathbb{E}(Z^\alpha) = \mathbb{E}\left(\left(\frac{1}{1+T}\right)^\alpha\right) = \mathbb{E}(1+T)^\mu = \int_0^1 z^\alpha f_Z(z)dz = 1/M\left(\alpha, \frac{1}{2}, \frac{c^2}{2}\right).$$

1. From Lin's results we conclude a sequence $\{\mathbb{E}(1+T)^{\mu_n} = \int_0^\infty (1+t)^{\mu_n} p(t)dt\}_{n=0}^\infty$, with $\mu_n = -\alpha_n \le 0$ in according with Lin's theorem, is enough to characterize $p(t)$.
2. For practical purposes only a finite set $\{\mu_n\}_{n=0}^N$ has to be taken into account. The maximum entropy method outlined in next section allows us to formulate an optimal criterion choice of $\mu_n$ based upon a convergence entropy theorem and previously formulated (Novi Inverardi and Tagliani, 2003, Appendix A).
3. The required value $N$ is calculated by imposing $p(t)$ and its approximation $p_N(t)$ has a preassigned error, which will be measured in terms of Kullback–Leibler distance, as outlined in the sequel.