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Testing variance parameters in models with a Kronecker product covariance structure

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Under a model having a Kronecker product covariance structure with compound symmetry, hypothesis testing for a correlation is investigated. Several tests are suggested and practical recommendations are made based on their type I error probabilities and powers.

1. Introduction

A multivariate model with a structured covariance matrix arises in many applications. Particularly, in some situations, data on a single subject may have a "matrix" or "tensor" structure (Naik and Rao, 2001; Singull et al., 2012). The scenario investigated in this article is one where on each subject, measurements are obtained on a set of p variables on q occasions, so that the random variable of interest is a $p \times q$ matrix. Suppose $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ is a random sample from the distribution of X, so that each $\mathbf{X}_i$ is a $p \times q$ matrix. We assume a matrix normal model having the following structure:

$$\mathbf{X}_i \sim N_{p,q}(\boldsymbol{\mu}, \boldsymbol{\Psi}, \boldsymbol{\Sigma}), \quad i = 1, \ldots, n, \tag{1}$$

where $\boldsymbol{\mu} : p \times q$ is an unstructured mean, $\boldsymbol{\Psi} : p \times p$ describes the covariance between the rows of $\mathbf{X}_i$, and $\boldsymbol{\Sigma} : q \times q$ is the corresponding covariance matrix between the columns, for $i = 1, \ldots, n$. The matrices $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$ and $\boldsymbol{\Psi}$ are unknown.

Model (1) with unstructured $\boldsymbol{\Sigma}$ and $\boldsymbol{\Psi}$ has been studied by many researchers (see Dutilleul, 1999; Lu and Zimmerman, 2005; Srivastava et al., 2008, for instance). In some circumstances, it will be reasonable to assume further structures on both $\boldsymbol{\Psi}$ and $\boldsymbol{\Sigma}$. For example, when the measurements are taken from p equivalent variables and q equivalent occasions, both $\boldsymbol{\Psi}$ and $\boldsymbol{\Sigma}$ will have a compound symmetry (CS) structure, also called uniform or intra-class covariance structure. Roy and Leiva (2008) provide an algorithm to compute the maximum likelihood estimates of the parameters when both $\boldsymbol{\Psi}$ and $\boldsymbol{\Sigma}$ have a CS structure, or an autoregressive of order one structure.

The present investigation is on testing hypotheses concerning the variance parameters when both $\boldsymbol{\Psi}$ and $\boldsymbol{\Sigma}$ have the CS structure. The correlation in $\boldsymbol{\Psi}$, i.e. the correlation between the p variables, is of particular interest. The model and this testing problem does arise in applications. Worsley et al. (1991) considered a trial where glucose measurements were taken

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from 10 subjects ($n = 10$) at different regions of their brains paired in both hemispheres ($q = 2$). It was of interest to test whether three specific pairs of the regions ($p = 3$) with equal distance are highly functionally related. This application will be used as an illustrative example later in this article.

The existing inference procedures under the model (1) are usually developed based on the likelihood function and the corresponding asymptotics. Our present investigation is on hypothesis testing concerning the correlations in the model. We shall use some higher order asymptotic procedures that modify the likelihood ratio test so as to achieve accurate performance in small samples; see Brazzale and Davison (2008). We shall suggest several modified tests based on the likelihood function and the restricted likelihood function. It will also be noted that two independent exact F tests exist for testing hypothesis concerning the correlations. We shall thus explore methodologies for combining the two F tests. The performance of all the tests will be evaluated based on simulated type I error probabilities and power, and recommendations will be made on the choice of the test for practical applications. The tests will be illustrated using the example mentioned earlier.

2. The model and the testing problem

The model (1) can equivalently be written as

$$\mathrm{vec}\,\mathbf{X}_i \sim N_{pq}(\mathrm{vec}\,\boldsymbol{\mu}, \boldsymbol{\Sigma} \otimes \boldsymbol{\Psi}), \quad i = 1, \ldots, n,$$

see Srivastava and Khatri (1979). Let $\mathbf{I}_p$ be the $p \times p$ identity matrix, $\mathbf{1}_p$ be the $p \times 1$ vector with all elements equal to 1 and $\mathbf{J}_p = \mathbf{1}_p\mathbf{1}_p'$. The CS structures for $\boldsymbol{\Sigma}$ and $\boldsymbol{\Psi}$ can be expressed as

$$\boldsymbol{\Sigma} = \sigma_1\left[(1-\rho_\Sigma)\mathbf{I}_q + \rho_\Sigma\mathbf{J}_q\right], \qquad \boldsymbol{\Psi} = \psi_1\left[(1-\rho_\psi)\mathbf{I}_p + \rho_\psi\mathbf{J}_p\right], \tag{2}$$

where σ_1, ψ_1, ρ_Σ and ρ_ψ are unknown parameters. For purpose of identifiability, we shall assume that $\psi_1 = 1$. Let $\boldsymbol{\Psi}^*$ denote the matrix $\boldsymbol{\Psi}$ under the restriction $\psi_1 = 1$. The distinct eigenvalues of $\boldsymbol{\Psi}^*$ and $\boldsymbol{\Sigma}$ are

$$\begin{cases}\lambda_1 &= 1 + (p-1)\rho_\psi \\ \lambda_2 &= 1 - \rho_\psi\end{cases} \quad \text{and} \quad \begin{cases}\lambda_3 &= \sigma_1(1 + (q-1)\rho_\Sigma) \\ \lambda_4 &= \sigma_1(1-\rho_\Sigma).\end{cases} \tag{3}$$

Let $\boldsymbol{\eta} = (\rho_\psi, \rho_\Sigma, \sigma_1)'$ be the vector of unknown variance parameters. We shall consider the following hypothesis testing problem concerning ρ_ψ:

$$H_0 : \rho_\psi = \rho_0 \quad \text{against} \qquad H_1 : -(p-1)^{-1} < \rho_\psi < 1, \quad \text{for a fixed } \rho_0. \tag{4}$$

3. Test procedures

We shall consider several possible tests for the hypotheses in (4), compare them numerically, and make practical recommendations.

3.1. Likelihood ratio tests

Let $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\eta}} = (\hat{\rho}_\psi, \hat{\rho}_\Sigma, \hat{\sigma}_1)'$ denote the MLEs of the respective parameters, and let $\hat{\boldsymbol{\eta}}_0 = (\rho_0, \hat{\rho}_{\Sigma 0}, \hat{\sigma}_{10})'$ denote the MLE of $\boldsymbol{\eta}$ under H_0. Furthermore, the MLE of $\boldsymbol{\mu}$ is the same under H_0 and otherwise. For testing the hypotheses in (4), the signed log-likelihood ratio statistic, say $r(\rho_0)$, is given by

$$r(\rho_0) = \mathrm{sign}(\hat{\rho}_\psi - \rho_0)\left[2\left\{\ell(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\eta}}) - \ell(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\eta}}_0)\right\}\right]^{1/2}, \tag{5}$$

where $\ell(\boldsymbol{\mu}, \boldsymbol{\eta})$ is the log-likelihood function, and $\mathrm{sign}(x)$ is $+1$ or -1 depending on whether $x > 0$ or $x < 0$, respectively.

Since the hypothesis of interest concerns a component of $\boldsymbol{\eta}$, a question of practical interest is whether one should use the likelihood of $\boldsymbol{\mu}$ and $\boldsymbol{\eta}$, or the restricted likelihood function of $\boldsymbol{\eta}$ alone. Let $\mathbf{P}_{1_p} = p^{-1}\mathbf{1}_p\mathbf{1}_p'$, $\mathbf{Q}_{1_p} = \mathbf{I}_p - \mathbf{P}_{1_p}$ and $\mathbf{X}_{ic} = \mathbf{X}_i - \bar{\mathbf{X}}$, where $\bar{\mathbf{X}}$ is the mean of the $\mathbf{X}_i$s. The restricted likelihood function of $\boldsymbol{\eta}$ can be obtained based on the distribution

$$\sum_{i=1}^{n} \mathrm{vec}\,\mathbf{X}_{ic}\mathrm{vec}'\mathbf{X}_{ic} \sim W_{pq}(n-1, \boldsymbol{\Sigma} \otimes \boldsymbol{\Psi}^*),$$

where $W_r(m, \Delta)$ denotes the r-dimensional central Wishart distribution with df = m and scale matrix Δ. Using the eigenspaces corresponding to the eigenvalues in (3), it can be shown that the minimal sufficient statistic for $\boldsymbol{\eta}$ is