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Portmanteau theorem for unbounded measures ☆

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Abstract

We prove an analogue of the portmanteau theorem on weak convergence of probability measures allowing measures which are unbounded on an underlying metric space but finite on the complement of any Borel neighbourhood of a fixed element.

Keywords: Weak convergence of bounded measures; Portmanteau theorem; Lévy measure

1. Introduction

Weak convergence of probability measures on a metric space has a very important role in probability theory. The well-known *portmanteau theorem* due to A.D. Alexandroff (see for example Theorem 11.1.1 in Dudley, 1989) provides useful conditions equivalent to weak convergence of probability measures; any of them could serve as the definition of weak convergence. Proposition 1.2.13 in the book of Meerschaert and Scheffler (2001) gives an analogue of the portmanteau theorem for bounded measures on $\mathbb{R}^d$. Moreover, Proposition 1.2.19 in Meerschaert and Scheffler (2001) gives an analogue for special unbounded measures on $\mathbb{R}^d$, more precisely, for extended real-valued measures which are finite on the complement of any Borel neighbourhood of $0 \in \mathbb{R}^d$.

By giving counterexamples we show that the equivalences of (c) and (d) in Propositions 1.2.13 and 1.2.19 in Meerschaert and Scheffler (2001) are not valid (see our Remarks 3 and 4). We reformulate Proposition 1.2.19 in Meerschaert and Scheffler (2001) in a more detailed form by adding new equivalent assertions to it (see Theorem 1). Moreover, we note that Theorem 1 generalizes the equivalence of (a) and (b) in Theorem 11.3.3 of Dudley (1989) in two aspects. On the one hand, the equivalence is extended allowing not necessarily finite measures which are finite on the complement of any Borel neighbourhood of a fixed element of an underlying metric space. On the other hand, we do not assume the separability of the underlying metric space to prove the equivalence. But we mention that this latter possibility is hiddenly contained in Problem 3, p. 312 in Dudley (1989). For completeness, we give a detailed proof of Theorem 1. Our proof goes along the lines of the proof

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of the original portmanteau theorem and differs from the proof of Proposition 1.2.19 in Meerschaert and Scheffler (2001).

To shed some light on the sense of a portmanteau theorem for unbounded measures, let us consider the question of weak convergence of infinitely divisible probability measures μ_n, $n \in \mathbb{N}$ towards an infinitely divisible probability measure μ_0 in case of the real line $\mathbb{R}$. Theorem VII.2.9 in Jacod and Shiryayev (1987) gives equivalent conditions for weak convergence $\mu_n \xrightarrow{w} \mu_0$. Among these conditions we have

$$\int_{\mathbb{R}} f \, \mathrm{d}\eta_n \to \int_{\mathbb{R}} f \, \mathrm{d}\eta_0 \quad \text{for all } f \in \mathscr{C}_2(\mathbb{R}), \tag{1}$$

where η_n, $n \in \mathbb{Z}_+$ are nonnegative, extended real-valued measures on $\mathbb{R}$ with $\eta_n(\{0\}) = 0$ and $\int_{\mathbb{R}} (x^2 \wedge 1) \, \mathrm{d}\eta_n(x) < \infty$ (i.e., Lévy measures on $\mathbb{R}$) corresponding to μ_n, and $\mathscr{C}_2(\mathbb{R})$ is the set of all real-valued bounded continuous functions f on $\mathbb{R}$ vanishing on some Borel neighbourhood of 0 and having a limit at infinity. Theorem 1 is about equivalent reformulations of (1) when it holds for all real-valued bounded continuous functions on $\mathbb{R}$ vanishing on some Borel neighbourhood of 0.

2. An analogue of the portmanteau theorem

Let $\mathbb{N}$ and $\mathbb{Z}_+$ be the set of positive and nonnegative integers, respectively. Let (X, d) be a metric space and x_0 be a fixed element of X. Let $\mathscr{B}(X)$ denote the σ-algebra of Borel subsets of X. A Borel neighbourhood U of x_0 is an element of $\mathscr{B}(X)$ for which there exists an open subset $\widetilde{U}$ of X such that $x_0 \in \widetilde{U} \subset U$. Let $\mathscr{N}_{x_0}$ denote the set of all Borel neighbourhoods of x_0, and the set of bounded measures on X is denoted by $\mathscr{M}^b(X)$. The expression "a measure μ on X" means a measure μ on the σ-algebra $\mathscr{B}(X)$.

Let $\mathscr{C}(X)$, $\mathscr{C}_{x_0}(X)$ and $\mathrm{BL}_{x_0}(X)$ denote the spaces of all real-valued bounded continuous functions on X, the set of all elements of $\mathscr{C}(X)$ vanishing on some Borel neighbourhood of x_0, and the set of all real-valued bounded Lipschitz functions vanishing on some Borel neighbourhood of x_0, respectively.

For a measure η on X and for a Borel subset $B \in \mathscr{B}(X)$, let $\eta|_B$ denote the restriction of η onto B, i.e., $\eta|_B(A) := \eta(B \cap A)$ for all $A \in \mathscr{B}(X)$.

Let μ_n, $n \in \mathbb{Z}_+$ be bounded measures on X. We write $\mu_n \xrightarrow{w} \mu$ if $\mu_n(A) \to \mu(A)$ for all $A \in \mathscr{B}(X)$ with $\mu(\partial A) = 0$. This is called *weak convergence of bounded measures* on X.

Now we formulate a portmanteau theorem for unbounded measures.

Theorem 1. *Let (X, d) be a metric space and x_0 be a fixed element of X. Let η_n, $n \in \mathbb{Z}_+$, be measures on X such that $\eta_n(X \setminus U) < \infty$ for all $U \in \mathscr{N}_{x_0}$ and for all $n \in \mathbb{Z}_+$. Then the following assertions are equivalent*:

(i) $\int_{X \setminus U} f \, \mathrm{d}\eta_n \to \int_{X \setminus U} f \, \mathrm{d}\eta_0$ *for all* $f \in \mathscr{C}(X)$, $U \in \mathscr{N}_{x_0}$ *with* $\eta_0(\partial U) = 0$,
(ii) $\eta_n|_{X \setminus U} \xrightarrow{w} \eta_0|_{X \setminus U}$ *for all* $U \in \mathscr{N}_{x_0}$ *with* $\eta_0(\partial U) = 0$,
(iii) $\eta_n(X \setminus U) \to \eta_0(X \setminus U)$ *for all* $U \in \mathscr{N}_{x_0}$ *with* $\eta_0(\partial U) = 0$,
(iv) $\int_X f \, \mathrm{d}\eta_n \to \int_X f \, \mathrm{d}\eta_0$ *for all* $f \in \mathscr{C}_{x_0}(X)$,
(v) $\int_X f \, \mathrm{d}\eta_n \to \int_X f \, \mathrm{d}\eta_0$ *for all* $f \in \mathrm{BL}_{x_0}(X)$,
(vi) *the following inequalities hold*:
  (a) $\limsup_{n \to \infty} \eta_n(X \setminus U) \leqslant \eta_0(X \setminus U)$ *for all open neighbourhoods* U of x_0,
  (b) $\liminf_{n \to \infty} \eta_n(X \setminus V) \geqslant \eta_0(X \setminus V)$ *for all closed neighbourhoods* V of x_0.

Proof. (i) $\Rightarrow$ (ii): Let U be an element of $\mathscr{N}_{x_0}$ with $\eta_0(\partial U) = 0$. Note $\eta_n|_{X \setminus U} \in \mathscr{M}^b(X)$, $n \in \mathbb{Z}_+$. By the equivalence of (a) and (b) in Proposition 1.2.13 in Meerschaert and Scheffler (2001), to prove $\eta_n|_{X \setminus U} \xrightarrow{w} \eta_0|_{X \setminus U}$ it is enough to check $\int_X f \, \mathrm{d}\eta_n|_{X \setminus U} \to \int_X f \, \mathrm{d}\eta_0|_{X \setminus U}$ for all $f \in \mathscr{C}(X)$. For this it suffices to show that for all real-valued bounded measurable functions h on X, for all $A \in \mathscr{B}(X)$ and for all $n \in \mathbb{Z}_+$ we have

$$\int_X h \, \mathrm{d}\eta_n|_A = \int_A h \, \mathrm{d}\eta_n. \tag{2}$$

By Beppo–Levi's theorem, a standard measure-theoretic argument implies (2).