Available online at www.sciencedirect.com
ScienceDirect
Statistics & Probability Letters 76 (2006) 1873–1881
STATISTICS & PROBABILITY LETTERS
www.elsevier.com/locate/stapro

Small ball probabilities for jump Lévy processes from the Wiener domain of attraction

Elena Shmileva[1]

Saint Petersburg Electrotechnical University, Professor Popov Str. 5, BM-2, St. Petersburg 197376, Russia

Received 25 August 2004; received in revised form 12 April 2006; accepted 24 April 2006
Available online 5 June 2006

Abstract

Let X_ρ be a jump Lévy process of intensity ρ which is close to the Wiener process if ρ is big. We study the behavior of shifted small ball probability, namely, $\mathbf{P}\{\sup_{t\in[0,1]}|X_\rho(t)-\lambda f(t)|\leqslant r\}$ under all possible relations between the parameters $r\to 0$, $\rho\to\infty$, $\lambda\to\infty$. The shift function f is of bounded variation of its derivative.

Keywords: Purely non-Gaussian Lévy process; Additive process; Small deviations; Skorokhod formula; Density transformation of Lévy processes

1. Problem statement

Let $(B[0,1],\ \|\cdot\|)$ be the space of bounded functions on $[0,1]$ endowed with the uniform norm. Introduce the purely non-Gaussian Lévy process determined by the generating triplet $(0,\gamma,\Lambda)$ (notations correspond to Sato, 1999). By Lévy–Ito integral representation it is

$$\xi(t)=\int_0^t\int_{\{|\ell|\leqslant 1\}}\ell\overline{\mathscr{P}}(\mathrm{d}s,\mathrm{d}\ell)+\int_0^t\int_{\{|\ell|>1\}}\ell\mathscr{P}(\mathrm{d}s,\mathrm{d}\ell)+\gamma t,$$

where $\mathscr{P}$ is the Poisson measure on $\mathrm{R}^+\times\mathrm{R}\backslash\{0\}$ associated to the deterministic measure $Leb\times\Lambda$. By $\overline{\mathscr{P}}$ we denote the centering of $\mathscr{P}$. In the sequel, by $\bar{\eta}$ we always denote the centering of a random element η.

Additionally assume that the support, $\mathrm{supp}(\Lambda)$, of the Lévy measure Λ is bounded and denote $\mathscr{L}:=\mathrm{supp}(\Lambda)$. This assumption, in particular, give us $\mathbf{E}\xi(t)<\infty$ and $\mathbf{D}\xi(t)=t\int_{\mathscr{L}}\ell^2\Lambda(\mathrm{d}\ell)<\infty$ for any $t\in[0,1]$. Denote $\sigma^2:=\mathbf{D}\xi(1)$.

Introduce the centered normalized process of intensity $\rho,\rho>0$

$$X_\rho(t):=\frac{\bar{\xi}(\rho\, t)}{\sqrt{\rho}},\quad t\in[0,1].$$

E-mail address: elena.shmileva@gmail.com.
[1]Research supported by RFBR-DFG Grant 04-01-04000 and by INTAS Grant 03-51-5018.

0167-7152/$ - see front matter
doi:10.1016/j.spl.2006.04.037

Since the variance of ξ is finite, the process belongs to the domain of attraction of the Wiener process, in other words, the weak invariance principle holds $X_\rho \overset{d}{\Rightarrow} \sigma W, \rho \to \infty$.

Our goal is to investigate the behavior of shifted small ball probabilities for the process X_ρ, i.e., we study $\mathbf{P}\{\|X_\rho - \lambda f\| < r\}$ as $r \to 0, \lambda \to \infty, \rho \to \infty$. We assume that the shift function f belongs to the space of admissible shifts for the Wiener process, that is

$$E := \left\{ f \in B[0,1] \mid f(0) = 0,\ f \in \mathrm{AC}[0,1],\ \int_0^1 f'(t)^2\,\mathrm{d}t < \infty \right\},$$

where AC[0, 1] is the space of absolutely continuous functions on [0, 1].

In the sequel, we need the following notations. For any $f \in B[0,1]$ put

$$|f|_E^2 := \begin{cases} \int_0^1 f'(t)^2\,\mathrm{d}t, & \text{when } f \in E, \\ \infty, & \text{when } f \notin \mathrm{E}, \end{cases}$$

this expression is often called *energy* of function f. By $B(f, r)$ denote the ball of center f and radius r in $(B[0,1], \|\cdot\|)$. We say that the maximal difference between the energy of f and the energy of any other function in the ball $B(f, r)$ is the *energy saving* in this ball and write

$$\Delta_f(r) := |f|_E^2 - \inf_{B(f,r)} |h|_E^2.$$

2. Main result

In the theorem below the shift function f obeys the condition $\mathrm{Var} f' < \infty$, though it is easy enough to obtain similar results for any regular function from E (see Shmileva, 2004). Nevertheless, for arbitrary admissible shift function the problem has not been solved.

Theorem 1. *Suppose $f \in E$ has a version of its Lebesgue derivative f' such that f' is a function of bounded variation* ($\mathrm{Var} f' < \infty$). *If the parameters λ, r, ρ satisfy*

(1) $\lambda \to \infty$, $\rho \to \infty$, $r/\lambda \to 0$,
(2) $\rho r^2 \to \infty$,
(3) $\rho/\lambda^2 \to \infty$,

then the following asymptotic estimates are true:

$$\mathbf{P}\{\|X_\rho - \lambda f\| < r\} \leqslant \exp\left\{ -\frac{\lambda^2}{2\sigma^2}|f|_E^2 - \frac{\pi^2\sigma^2}{8r^2}(1 + \mathrm{o}(1)) + \frac{\lambda r}{\sigma^2}(|f'(1)| + \mathrm{Var} f')(1 + \mathrm{o}(1)) + \frac{1}{6\sigma^6}\frac{\lambda^3}{\rho^{1/2}}\left(\theta_f + \mathrm{o}(1)\right) \right\},$$

$$\mathbf{P}\{\|X_\rho - \lambda f\| < r\} \geqslant \exp\left\{ -\frac{\lambda^2}{2\sigma^2}|f|_E^2 - \frac{\pi^2\sigma^2}{8r^2(1-\delta)^2}(1 + \mathrm{o}(1)) + (2\delta - 1)\frac{\lambda r}{\sigma^2}(|f'(1)| + \mathrm{Var} f')(1 + \mathrm{o}(1)) + \frac{1}{6\sigma^6}\frac{\lambda^3}{\rho^{1/2}}\left(\theta_f + \mathrm{o}(1)\right) \right\},$$

where $\delta \in [0, 1)$ is arbitrary and $\theta_f := \int_{\mathscr{L}} \ell^3 \Lambda(\mathrm{d}\ell) \cdot \int_0^1 f'^3(t)\,\mathrm{d}t$.

3. History and comments

Let us compare Theorem 1 with previous results. The shifted small ball estimate of the Wiener process was obtained by Grill (1991)

$$\mathbf{P}\{\|W - \lambda f\| < r\} = \exp\left\{ -\frac{\lambda^2}{2}|f|_E^2 - \frac{\pi^2}{8r^2}(1 + \mathrm{o}(1)) + \mathrm{R} \right\} \quad \text{as } \lambda \to \infty,\ r \to 0,$$