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Superposition of renewal processes and an application to multi-server queues ☆

Offer Kella[a,*], Wolfgang Stadje[b]

[a]Department of Statistics, The Hebrew University of Jerusalem, Mount Scopus, Jerusalem 91905, Israel
[b]Department of Mathematics and Computer Science, University of Osnabrück, 49069 Osnabrück, Germany

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Abstract

The aim of this paper is to compare the waiting times of customers in multiple-server queues, where the idle times are removed, with different numbers of servers. For this purpose we develop some results regarding the vector-valued marked point process whose points are arrival epochs of the superposition of renewal processes with different continuous inter-arrival distribution and the marks are the vectors of forward recurrence times of the various renewal processes at these arrival epochs.

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1. Introduction

We consider the point process generated by the superposition of K independent renewal processes on $(0,\infty)$. For $i=1,\ldots,K$ let $N^i=\{N^i(t)\mid t\geqslant 0\}$ be the ith renewal process; its points are denoted by $S^i_n=\inf\{t\geqslant 0\mid N^i(t)=n\}$, $n\in\mathbb{Z}_+$, and its forward recurrence times by $\gamma_i(t)=S^i_{N^i(t)+1}-t$, $t\geqslant 0$. The inter-arrival times of N^i have a continuous distribution function F_i and finite mean μ_i. The assumption that F_i is continuous for all i is made in order to avoid the nuisance of having multiple renewal points (see Alsmeyer, 1996, for a discussion of this point). Let T_n, $n\geqslant 1$, be the nth point of the superposition of $N^1,\ldots,N^K$, set $T_0=0$ and define the forward recurrence time sequence $Y=\{Y^n\mid n\in\mathbb{Z}_+\}$ by $Y^n=(\gamma_1(T_n),\ldots,\gamma_K(T_n))$. Clearly, Y is a discrete-time Markov process and can be started by any initial distribution π for Y^0. $\mathbb{P}_\pi$ and $\mathbb{E}_\pi$ denote probability and expectation under $Y^0\sim\pi$. Motivated by an application to multi-server queues, we want to derive asymptotic characteristics of T_n and Y^n.

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*Corresponding author. Fax: +972 2 5882839.
E-mail addresses: Offer.Kella@huji.ac.il (O. Kella), wolfgang@mathematik.uni-osnabrueck.de (W. Stadje).

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For a positive random variable X having a finite mean we denote by X^e a random variable having the stationary forward recurrence time distribution of X, that is, X^e has the density $t \mapsto \mathbb{P}[X>t]/\mathbb{E}X$. For any distribution function (d.f.) F, we let $\bar{F}(t) = 1 - F(t)$. If $F(0) = 0$ and $\mu = \int_0^\infty x F(\mathrm{d}x) < \infty$, the corresponding equilibrium d.f. is

$$F^e(t) = \frac{1}{\mu} \int_0^t \bar{F}(x)\,\mathrm{d}x.$$

The following results on T_n and Y^n are proved in Section 2:

(a) A stationary distribution for Y is

$$\pi_0(\mathrm{d}x_1, \ldots, \mathrm{d}x_K) = \sum_{i=1}^K \frac{\mu_i^{-1}}{\sum_{k=1}^K \mu_k^{-1}} F_i(\mathrm{d}x_i) \prod_{k \neq i} F_k^e(\mathrm{d}x_k). \tag{1.1}$$

(b) When the initial condition is π_0, then

$$\mathbb{E}_{\pi_0} T_n = n \Big/ \sum_{k=1}^K \mu_k^{-1}. \tag{1.2}$$

(c) If $F_1, \ldots, F_K$ are spread out, then π_0 is the unique stationary distribution and is the limiting distribution of Y_n for any initial condition (possibly random).

(d) For any initial distribution π,

$$\frac{T_n}{n} \to \left(\sum_{k=1}^K \mu_k^{-1} \right)^{-1} \quad \text{as } n \to \infty \ \mathbb{P}_\pi\text{-almost surely}, \tag{1.3}$$

and if in addition one of the marginals of π has a finite mean then

$$\frac{\mathbb{E}_\pi T_n}{n} \to \left(\sum_{k=1}^K \mu_k^{-1} \right)^{-1} \quad \text{as } n \to \infty. \tag{1.4}$$

We note that versions of (a) and (c) are stated in Alsmeyer (1996) for the case of the age rather than the forward renewal processes.

In Section 3 we consider the special case $F_1 = \cdots = F_K = \mathrm{Erl}(m, \lambda)$ (the Erlang distribution with parameters $m \in \mathbb{N}$ and $\lambda > 0$). Let $v_{2,n}, \ldots, v_{K,n}$ be the $K-1$ forward recurrence times $\gamma_i(T_n)$ at time T_n of those renewal processes to which the point T_n does not belong. It follows from results (a) and (c) that $(v_{2,n}, \ldots, v_{K,n})$ has the asymptotic distribution M which is the product of $K-1$ equilibrium distributions for $\mathrm{Erl}(m, \lambda)$. We show that this convergence is exponential in the following sense: for any Borel function f: $\mathbb{R}_+^{K-1} \to \mathbb{R}_+$ satisfying $\int f\,\mathrm{d}M < \infty$ we have

$$\left| \mathbb{E}_f(v_{2,n}, \ldots, v_{K,n}) - \int f\,\mathrm{d}M \right| \leqslant (m^{K-1} - 1) \left(\int f\,\mathrm{d}M \right) \alpha(m, K)^{nm-1},$$

where $\alpha(m, K) \in (0, 1)$ is a certain constant.

These renewal-theoretic results are needed in a problem about a *multi-server queue without idleness*. Consider a service system composed of K identical servers working in parallel. The service times $X_1, X_2, \ldots$ of the customers are assumed to be i.i.d. positive random variables with the common distribution function F. The system starts operating at time 0 and is immediately occupied by the first K customers. We suppose that all servers will always be supplied with work such that none will ever be idle. This is for example the case if (i) infinitely many customers are already waiting at time 0 or (ii) a new customer is called in every time nobody is waiting and one of the servers becomes empty. In this setting the starting times of services in $(0, \infty)$ form a superposition of K i.i.d. renewal processes whose nth point T_n is the departure time of the nth customer.