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Gnedenko–Raikov's theorem, central limit theory, and the weak law of large numbers

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Abstract

This note is devoted to the connection between a theorem due to Gnedenko, classical central limit theory, and the weak law of large numbers.

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1. Introduction

The aim of this note is to discuss the connection between the i.i.d. version of a theorem due to Gnedenko (1939), which, in turn extends Raikov's theorem (1938), classical central limit theory and an application of a recent extension of the Kolmogorov–Feller weak law of large numbers (Gut, 2004), see also Gut (2005, Theorem 6.4.2).

1.1. Background

Gnedenko's theorem (1939), or Gnedenko and Kolmogorov (1968, Section 28, Theorem 5) states that the row sums of independent, infinitesimal random variables in a triangular array are asymptotically normal if and only if the sum of the squares of the row elements, centered at truncated means, is relatively stable. Raikov (1938), see also Gnedenko and Kolmogorov (1968, Section 28, Theorem 4), assumes finite means and centering at expectations.

The following variant for sums of i.i.d. random variables is (Giné et al., 1977, Lemma 3.2):

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Theorem 1.1. *Let $X, X_1, X_2, \ldots$ be i.i.d. random variables with mean* 0, *and set* $S_n = \sum_{k=1}^{n} X_k$, $n \geqslant 1$. *Furthermore, suppose that* $\{\sigma_n, n \geqslant 1\}$, *is a sequence of positive reals increasing to* $+\infty$, *and set*

$$T_n = \sum_{k=1}^{n} X_k^2.$$

If $X \in \mathscr{D}(N)$, *that is, if*

$$\frac{S_n}{\sigma_n} \xrightarrow{\mathrm{d}} N(0,1) \quad \text{as } n \to \infty, \tag{1.1}$$

then

$$\frac{T_n}{\sigma_n^2} \xrightarrow{\mathrm{p}} 1 \quad \text{as } n \to \infty. \tag{1.2}$$

The purpose of this note is

(a) to prove that (1.1) $\iff$ (1.2),
(b) that the implications do not remain true when X belongs to the domain of attraction of a stable law with index $\alpha \in (0, 2) - X \in \mathscr{D}(\alpha)$, $0 < \alpha < 2$.

The main point of this note is that the proofs do not involve any computations, rather, the conclusions are obtained via a series of implications, more precisely, by joining facts from the theory of domains of attraction and a recent generalization of the Kolmogorov–Feller weak law of large numbers, (Gut, 2004) and also Gut (2005, Theorem 6.4.2).

2. Preliminaries

For facts about domains of attraction and central limit theory we refer to Gnedenko and Kolmogorov (1968), Feller (1971) or to Gut (2005) without explicit mention along the way. For the readers' convenience we include the following definition; see Gut (2005), Appendix and further references given there.

Definition 2.1. Let $a > 0$. A positive measurable function u on $[a, \infty)$ *varies regularly* at infinity with exponent ρ, $-\infty < \rho < \infty$, denoted $u \in \mathscr{RV}(\rho)$, iff

$$\frac{u(tx)}{u(t)} \to x^{\rho} \quad \text{as } t \to \infty \text{ for all } x > 0.$$

If $\rho = 0$ the function is *slowly varying* at infinity; $u \in \mathscr{SV}$.

Typical examples of regularly varying functions are x^{ρ}, $x^{\rho} \log^{+} x$, $x^{\rho} \log^{+} \log^{+} x$, and so on. Typical slowly varying functions are the same with $\rho = 0$.

Now, from central limit theory we know that X belongs to the domain of attraction of the normal distribution—$X \in \mathscr{D}(N)$—iff the sequence of truncated variances is slowly varying. Technically, let $\{\sigma_n,\ n \geqslant 1\}$, be a sequence of positive reals increasing to $+\infty$. Then

$$\frac{S_n}{\sigma_n} \xrightarrow{\mathrm{d}} \mathrm{N}(0,1) \quad \text{as } n \to \infty \iff U(x) = EX^2 I\{|X| \leqslant x\} \in \mathscr{SV}. \tag{2.1}$$

Moreover,

$$\sigma_n \in \mathscr{RV}(1/2) \quad \text{as } n \to \infty, \tag{2.2}$$

$$a_n := \frac{nU(\sigma_n)}{\sigma_n^2} \to 1 \quad \text{as } n \to \infty. \tag{2.3}$$