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On uniform nonintegrability for a sequence of random variables

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ABSTRACT

In this correspondence, we introduce the notion of a sequence of random variables being uniformly nonintegrable. Sufficient and, separately, necessary conditions for uniform nonintegrability are presented and we also establish two equivalent characterizations of uniform nonintegrability, one of which is a uniform nonintegrability analogue of the celebrated de La Vallée Poussin criterion for uniform integrability. Several illustrative examples are presented.

1. Introduction

We introduce the notion of a sequence of random variables being *uniformly nonintegrable* and we present as our main results:

(i) sufficient and, separately, necessary conditions for this notion and
(ii) two equivalent characterizations of this notion.

All random variables under consideration are defined on a fixed but otherwise arbitrary probability space $(\Omega, \mathcal{F}, P)$. We shall replace the usual notation $E(XI_A)$ by $E(X : A)$.

To motivate our investigation, we recall some definitions and results concerning integrability and uniform integrability of random variables. A random variable X is said to be *integrable* if $E|X| < \infty$. It follows from the Lebesgue dominated convergence theorem that X is integrable if and only if $\lim_{a\to\infty} E(|X| : |X| > a) = 0$ or, equivalently, $\inf_{N\geq 1} E(|X| : |X| > N) = 0$. This leads to the following definition of uniform integrability.

Definition 1.1. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be *uniformly integrable* (UI) if $\lim_{a\to\infty} \sup_{n\geq 1} E(|X_n| : |X_n| \geq a) = 0$ or, equivalently, $\inf_{N\geq 1} \sup_{n\geq 1} E(|X_n| : |X_n| \geq N) = 0$.

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The *uniform integrability criterion* (see, e.g., Chow and Teicher, 1997, p. 94, or Chung, 1974, p. 96) asserts that a sequence of random variables $\{X_n, n \geq 1\}$ is UI if and only if

$$\sup_{n\geq 1} E|X_n| < \infty \tag{1.1}$$

and

$$\text{for all } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that for every } A \in \mathcal{F} \text{ with } P(A) < \delta,$$
$$\text{one has}$$
$$\sup_{n\geq 1} E(|X_n| : A) < \varepsilon. \tag{1.2}$$

It is easy to construct examples showing that (1.1) and (1.2) are independent in the sense that neither implies the other. Hu and Rosalsky (2011) showed that (1.2) is indeed equivalent to the apparently stronger condition

$$\text{for all } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that for every sequence}$$
$$\{A_n, n \geq 1\} \text{ in } \mathcal{F} \text{ with } \sup_{n\geq 1} P(A_n) < \delta, \text{ one has } \sup_{n\geq 1} E(|X_n| : A_n) < \varepsilon.$$

The following classical result of the renowned Belgian mathematician Charles de La Vallée Poussin (see Meyer, 1966, p. 19) provides another characterization of uniform integrability. We refer to it as the *de La Vallée Poussin criterion for UI.*

Theorem 1.1 (*de La Vallée Poussin*)**.** *A sequence of random variables $\{X_n, n \geq 1\}$ is UI if and only if there exists a convex non-decreasing function $G : [0, \infty) \to [0, \infty)$ with $G(0) = 0$ such that*

$$\lim_{x\to\infty} \frac{G(x)}{x} = \infty \quad \textit{and} \quad \sup_{n\geq 1} E(G(|X_n|)) < \infty.$$

For the sufficiency half, the condition that G is a convex non-decreasing function defined on $[0, \infty)$ with $G(0) = 0$ can be weakened to the condition that G is a nonnegative Borel measurable function defined on $[0, \infty)$.

Chong (1979), Klenke (2014, p. 138), and Chandra (2015) provided different proofs of the de La Vallée Poussin criterion for UI.

For a random variable X, we of course say that X is *nonintegrable* (NI) if $E|X| = \infty$. It follows from the Lebesgue monotone convergence theorem that

$$X \text{ is NI if and only if } \lim_{a\to\infty} E(|X| : |X| \leq a) = \infty. \tag{1.3}$$

In view of Definition 1.1 and (1.3), a natural way to define uniform nonintegrability for a sequence of random variables is provided by the following definition.

Definition 1.2. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be *uniformly nonintegrable* (UNI) if

$$\lim_{a\to\infty} \inf_{n\geq 1} E(|X_n| : |X| \leq a) = \infty \tag{1.4}$$

or, equivalently,

$$\sup_{N\geq 1} \inf_{n\geq 1} E(|X_n| : |X_n| \leq N) = \infty.$$

Remark 1.1. If $\{X_n, n \geq 1\}$ is UNI, then X_n is NI for all $n \geq 1$. To see this, let $n_0 \geq 1$. Then for $a > 0$,

$$E|X_{n_0}| \geq E(|X_{n_0}| : |X_{n_0}| \leq a) \geq \inf_{n\geq 1} E(|X_n| : |X_n| \leq a) \to \infty \quad \text{as } a \to \infty.$$

In Section 2 we present two preliminary lemmas and in Section 3 we state and prove the main results. Illustrative examples are presented in Section 4. Section 5 contains some concluding remarks.

We end this section by remarking that the full significance of the concept of a sequence of random variables being UNI is not completely clear at the present stage, but some useful conclusions of a negative nature are illustrated in Section 5. These conclusions are not true in general for a sequence of NI random variables. This is not at all surprising since UNI, in contradistinction to UI which is a positive type of concept and hence is able to produce positive results such as $\mathcal{L}_p$ convergence for a sequence of random variables, is the antithesis of UI and should naturally yield results of the type to be presented in Section 5.

2. Preliminary lemmas

The following lemmas will be used in Section 3.