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On bounding the union probability using partial weighted information[☆]

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ABSTRACT

Lower bounds on the finite union probability are established in terms of the individual event probabilities and a weighted sum of the pairwise event probabilities. The lower bounds have at most pseudo-polynomial computational complexity and generalize recent analytical bounds.

1. Introduction

Lower and upper bounds on the union probability $P\left(\bigcup_{i=1}^{N} A_i\right)$ in terms of the individual event probabilities $P(A_i)$'s and the pairwise event probabilities $P(A_i \cap A_j)$'s have been actively investigated in the recent past. Optimal bounds can be obtained numerically by solving linear programming (LP) problems with 2^N variables (for instance, see Prékopa and Gao, 2005; Veneziani, unpublished). Since the number of variables is exponential in the number of events, N, some suboptimal but numerically efficient bounds have been proposed, such as the algorithmic Bonferroni-type bounds in Kuai et al. (2000b) and Behnamfar et al. (2005).

Among the established analytical bounds is the Kuai–Alajaji–Takahara lower bound (for convenience, hereafter referred to as the KAT bound) (Kuai et al., 2000a) that was shown to be better than the Dawson–Sankoff (DS) (Dawson and Sankoff, 1967) and the D. de Caen (DC) bounds (De Caen, 1997). Noting that the KAT bound is expressed in terms of $\{P(A_i)\}$ and only the *sums* of the pairwise event probabilities, i.e., $\{\sum_{j:j\neq i} P(A_i \cap A_j)\}$, in order to fully exploit all pairwise event probabilities, it is observed in Behnamfar et al. (2007), Hoppe (2006) and Hoppe (2009) that the analytical bounds can be further improved algorithmically by optimizing over subsets. Furthermore, in Prékopa and Gao (2005), the KAT bound is extended by using additional partial information such as the sums of joint probabilities of three events, i.e., $\{\sum_{j,l} P(A_i \cap A_j \cap A_l),\ i = 1, \ldots, N\}$. Recently, using the same partial information as the KAT bound, i.e., $\{P(A_i)\}$ and $\{\sum_{j:j\neq i} P(A_i \cap A_j)\}$, the optimal lower/upper bound and a new analytical bound which is sharper than the KAT bound were developed by Yang–Alajaji–Takahara in Yang et al. (2014) and Yang et al. (2014) (these two bounds are respectively referred to as the YAT-I and YAT-II bounds).

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In this work, we extend the existing analytical lower bounds, the KAT and YAT-II bounds, and establish two new classes of lower bounds on $P\left(\bigcup_{i=1}^{N} A_i\right)$ using $\{P(A_i)\}$ and $\{\sum_j c_j P(A_i \cap A_j)\}$ for a given weight or parameter vector $\boldsymbol{c} = (c_1, \ldots, c_N)^T$. These lower bounds are shown to have at most pseudo-polynomial computational complexity and to be sharper in certain cases than the Gallot–Kounias (GK) (Gallot, 1966; Kounias, 1968) and Prékopa–Gao (PG) bounds (Prékopa and Gao, 2005) even though the latter bounds employ more information on the events joint probabilities.

More specifically, we first propose a novel expression for the union probability given a weight vector $\boldsymbol{c}$. Using the Cauchy–Schwarz inequality, several existing bounds, such as the bound in Cohen and Merhav (2004), and the DC and GK bounds, can be directly derived from this new expression. Next, we derive two new classes of lower bounds as functions of the weight vector $\boldsymbol{c}$ by solving linear programming problems. The KAT and YAT-II analytical bounds are shown to be special cases of the new classes of lower bounds. Furthermore, it is noted that the proposed lower bounds can be sharper than the GK bound under some conditions.

We emphasize that our bounds can be applied to any general estimation problem involving the probability of a finite union of events. In particular, they can be applied to effectively estimate and analyze the error performance of communication systems (e.g., see Yang et al., 2014, Kuai et al., 2000b, Behnamfar et al., 2007, Cohen and Merhav, 2004, Seguin, 1998, Mao et al., 2013). Such bounds are also pertinently useful in the analysis of asymptotic problems such as the Borel–Cantelli lemma and its generalization (Erdős and Rényi, 1959; Feng et al., 2009; Frolov, 2012; Feng and Li, 2013). Finally, we note that the proposed bounds provide useful tools for chance-constrained stochastic programs (e.g., see Prékopa, 1995, Shapiro et al., 2014) in operations research. More specifically, using partial information of uncertainty, the proposed bounds on the union probability can be applied to formulate tractable conservative approximations of chance-constrained stochastic problems, which can be solved efficiently and produce feasible solutions for the original problems (see, for instance, Pintér, 1989, Nemirovski and Shapiro, 2006, Ben-Tal et al., 2009). An example of such application is the work in Ahmed and Papageorgiou (2013) on the probabilistic set covering problem with correlations, where the KAT bound is used for the case where only partial information on the correlation is available.

The outline of this paper is as follows. In Section 2, we propose a new expression of the union probability using weight vector $\boldsymbol{c}$ such that many existing bounds can be directly derived from this expression. In Section 3, we develop two new classes of lower bounds as functions of the weight vector $\boldsymbol{c}$ and discuss their connection with existing bounds, including the KAT, YAT-II and GK bounds. As by-products of the new lower bounds, two new classes of upper bounds are also obtained. Finally, in Section 4, we compare via numerical examples existing lower bounds with the proposed bounds under different choices of weight vectors.

2. A new expression of the union probability

For simplicity, and without loss of generality, we assume that the events $\{A_1, \ldots, A_N\}$ are in a finite probability space $(\Omega, \mathscr{F}, P)$, where N is a fixed positive integer. Let $\mathscr{B}$ denote the collection of all non-empty subsets of $\{1, 2, \ldots, N\}$. Given $B \in \mathscr{B}$, we let ω_B denote the atom in $\cup_{i=1}^{N} A_i$ such that for all $i = 1, \ldots, N$, $\omega_B \in A_i$ if $i \in B$ and $\omega_B \notin A_i$ if $i \notin B$ (note that some of these "atoms" may be the empty set). For ease of notation, for a singleton $\omega \in \Omega$, we denote $P(\{\omega\})$ by $p(\omega)$ and $P(\omega_B)$ by p_B. Since $\{\omega_B : i \in B\}$ is the collection of all the atoms in A_i, we have $P(A_i) = \sum_{\omega \in A_i} p(\omega) = \sum_{B \in \mathscr{B}: i \in B} p_B$, and

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{B \in \mathscr{B}} p_B. \tag{1}$$

Suppose there are N functions $f_i(B)$, $i = 1, \ldots, N$ such that $\sum_{i=1}^{N} f_i(B) = 1$ for any $B \in \mathscr{B}$. If we further assume that $f_i(B) = 0$ if $i \notin B$, we can write

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{B \in \mathscr{B}} \left(\sum_{i=1}^{N} f_i(B)\right) p_B = \sum_{i=1}^{N} \sum_{B \in \mathscr{B}: i \in B} f_i(B) p_B. \tag{2}$$

Note that if we define the degree of a subset $A \subset \Omega$, $\deg(A)$, to be the number of A_i's that contain A, then by the definition of ω_B, we have $\deg(\omega_B) = |B|$. Therefore,

$$f_i(B) = \begin{cases} \dfrac{1}{|B|} = \dfrac{1}{\deg(\omega_B)} & \text{if } i \in B \\ 0 & \text{if } i \notin B \end{cases} \tag{3}$$

satisfies $\sum_{i=1}^{N} f_i(B) = 1$ and (2) becomes

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} \sum_{B \in \mathscr{B}: i \in B} \frac{p_B}{\deg(\omega_B)} = \sum_{i=1}^{N} \sum_{\omega \in A_i} \frac{p(\omega)}{\deg(\omega)}. \tag{4}$$

Note that many of the existing bounds, such as the DC bound, the KAT bound and the recent bounds in Yang et al. (2014) and Yang et al. (2014), are based on (4).