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Characteristic function of time-inhomogeneous Lévy-driven Ornstein–Uhlenbeck processes

Frédéric Vrins

Louvain School of Management (LSM), Université catholique de Louvain, Chaussée de Binche 151, 7000 Mons, Belgium
Center for Operations Research and Econometrics (CORE), Université catholique de Louvain, Voie du Roman Pays 34, 1348 Louvain-la-Neuve, Belgium

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ABSTRACT

We derive the characteristic function (CF) of integrals of Lévy-driven Ornstein–Uhlenbeck processes with time-inhomogeneous coefficients. The resulting expression takes the form of the exponential integral of the time-changed characteristic exponent. This result is applied to some examples leading to closed form expressions. In particular, it drastically simplifies the calculations of the CF of integrated Compound Poisson processes compared to the standard approach relying on joint conditioning on inter-arrival jump times.

1. Introduction

We consider a time-inhomogeneous Ornstein–Uhlenbeck (OU) process λ governed by a Background Driving Lévy Process (BDLP) X:

$$d\lambda_t = (\alpha(t) - \beta(t)\lambda_t)dt + \sigma(t)dX_t. \tag{1}$$

The Lévy triple of X is noted (μ, σ, ν) where $\nu(z)$ stands for the density of the Lévy measure. These processes are called *generalized OU processes*. They have been studied in Barndorff-Nielsen and Shepard (2001) in the context of stochastic volatility modeling in econometrics and finance in the special case of constant coefficients.

In this letter, we are interested in the characteristic function $\varphi_{\Lambda_{s,t}}(x)$ of the path integral $\Lambda_{s,t} = \int_s^t \lambda_u du$ conditional upon the σ-field $\mathcal{F}_s$ standing for the filtration up to time s:

$$\varphi_{\Lambda_{s,t}}(x) := \mathrm{E}[e^{ix\Lambda_{s,t}} | \mathcal{F}_s] := \mathrm{E}_s[e^{ix\Lambda_{s,t}}]. \tag{2}$$

Integrals of Markov processes are popular stochastic processes as a result of their numerous applications. In finance for instance, these integrals appear in Asian options (Geman and Yor, 1993; Carr and Schroder, 2004) and their Laplace transforms are appealing when explicit formula for the distributions are too involved or cannot be found. In some cases however, these characteristic functions *naturally* embed key information. For instance, $\varphi_{\Lambda_{s,t}}(i)$ is a key calibration condition when dealing with interest rates in short-rate models or survival probability distributions in Cox models (see e.g. Brigo and Mercurio, 2006, Lando, 2004, Bielecki et al., 2011).

Closed form expressions of the Laplace transforms of path integrals have been derived in Pollett and Stefanov (2002) for continuous-time Markov chains taking value on the set of positive integers. With regards to diffusion processes, very general results have been obtained for affine, quadratic and geometric models. The corresponding results are derived from

E-mail address: frederic.vrins@uclouvain.be.

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the transition probabilities and the property that the infinitesimal generator is dependent on the space variable only, i.e. the stochastic differential equations (SDE) have time-homogeneous coefficients. Explicit formulas for many standard processes including the integrated Brownian Ornstein–Uhlenbeck, Square-Root and Jacobi diffusions are available in Albanese and Lawi (2005) and Hurd and Kuznetsov (2008). The specific case of integral of geometric Brownian motion, closely linked to squared Bessel processes, has extensively been studied in Yor (1992).

In spite of this extensive literature, some important results are still lacking for interesting stochastic processes, including the integral of generalized time-inhomogeneous OU processes. In this letter, we provide a simple formula for getting those characteristic functions based on time-changed integrals of the Lévy characteristic exponent. We illustrate the interest of the results on some examples.

2. Characteristic function of $\Lambda_{s,t}$

The logarithm of the characteristic function $\varphi_{X_t}(x)$ of a Lévy process X satisfies

$$\psi_{X_t}(x) := \ln \varphi_{X_t}(x) = t \ln \varphi_{X_1}(x) = t\psi_X(x) \tag{3}$$

where

$$\psi_X(x) := \psi_{X_1}(x) = ix\mu - \frac{1}{2}\sigma x^2 + \int_{\mathbb{R}} (\mathrm{e}^{ixz} - 1 - ixz\, 1_{\{|z|<1\}})\nu(dz) \tag{4}$$

is the *characteristic exponent* of X (see e.g. Cont and Tankov, 2004).

The proposition below gives the characteristic exponent of a stochastic integral with respect to a Lévy process.

Proposition 1. *Let X be a Lévy process with characteristic exponent ψ_X and define the semi-martingale $Y_{s,t} = \int_s^t \sigma(u)dX_u$ for some deterministic integrable function σ. Then,*

$$\psi_{Y_{s,t}}(x) = \int_s^t \psi_X(\sigma(u)x)du$$

where $\psi_{Y_{s,t}}(x) := \ln \varphi_{Y_{s,t}}(x)$.

Proof. See Appendix A.1. □

Observing that the bounds (s, t) in $Y_{s,t}$ are fixed, the above proposition holds even for integrands of the form $\sigma(s, t, u)$.

Proposition 2. *Let λ be the solution to Eq. (1). Then, $\varphi_{\Lambda_{s,t}}(x) = \exp\{\psi_{\Lambda_{s,t}}(x)\}$ where*

$$\psi_{\Lambda_{s,t}}(x) = ixM(s,t) + \int_s^t \psi_X\left(xK(u,t)\right) du \tag{5}$$

with

$$G(s,t) := \mathrm{e}^{-\int_s^t \beta(u)du}, \qquad I(s,t) := \int_s^t \alpha(u)G(u,s)du, \qquad K(s,t) := \int_s^t \sigma(s)G(s,u)du$$

$$M(s,t) := (t-s)\lambda_s + \int_s^t (t-u)\left(\alpha(u) - \beta(u)G(s,u)(\lambda_s + I(s,u))\right) du.$$

Proof. See Appendix A.2. □

The above result can be applied to obtain the characteristic function of the integral $\int_s^t f(u)\lambda_u du$ of a process λ solving (1) weighted in a time-dependent way by a sufficiently regular function f. Indeed, the stochastic process $\tilde{\lambda}_t := f(t)\lambda_t$ is a solution to (1) with $\tilde{\lambda}_s = f(s)\lambda_s$, $\alpha(t) \leftarrow f(t)\alpha(t)$, $\beta(t) \leftarrow \beta(t) - f'(t)/f(t)$ and $\sigma(t) \leftarrow f(t)\sigma(t)$.

Example 1. Let $X = W$ be a Brownian motion. Then,

$$\int_s^t \psi_X(K(u,t)x)du = -\int_s^t \frac{(xK(u,t))^2}{2}du = -\frac{x^2}{2}V(s,t).$$

Therefore,

$$\psi_{\Lambda_{s,t}}(x) = ixM(s,t) - \frac{x^2}{2}V(s,t) \tag{6}$$

where $V(s,t) := \int_s^t K^2(u,t)du$ is the variance of the Gaussian random variable $\Lambda_{s,t}$.

The next example is a drifted Brownian bridge, whose expectation is $\gamma t(T-t)$.