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Artifactual unit root behavior of Value at risk (VaR)

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ABSTRACT

An effective model for time-varying quantiles of a time series is of considerable practical importance across various disciplines. In particular, in financial risk management, computation of Value-at-risk (VaR), one of the most popular risk measures, involves knowledge of quantiles of portfolio returns. This paper examines the random walk behavior of VaRs constructed under two most common approaches, viz. historical simulation and the parametric approach using GARCH models. We find that sequences of historical VaRs appear to follow a unit root model, which can be an artifact under some settings, whereas its counterpart constructed via the parametric approach does not follow a random walk model by default.

1. Introduction

Forecasting future values of an observed time series is an important problem for a wide range of disciplines, including genetics, medical studies, meteorology, financial investments and risk management. There has been extensive literature discussing quantile regression; see Koenker (2005), Koenker and Xiao (2004) and Koenker and Xiao (2006) among others in which quantiles of autoregressive models are studied. In particular, Koenker and Xiao (2004) consider a quantile autoregression (QAR) model as follows: Let $\{r_t\}$ be a first-order autoregressive process that satisfies:

$$r_t = \phi r_{t-1} + u_t, \tag{1}$$

where u_t denotes white noise, one can write the τth conditional quantile of r_t as

$$Q_{r_t}(\tau \mid r_{t-1}) = \phi r_{t-1} + Q_u(\tau), \tag{2}$$

where $Q_X(\tau \mid \mathcal{F})$ denotes the τth conditional quantile of X given filtration $\mathcal{F}$. Observe that

$$\begin{aligned} Q_{r_t}(\tau \mid r_{t-1}) - Q_{r_{t-1}}(\tau \mid r_{t-2}) &= \{\phi r_{t-1} + Q_u(\tau)\} - \{\phi r_{t-2} + Q_u(\tau)\} \\ &= \phi(\tau)(r_{t-1} - r_{t-2}) \\ &= \cdots = \sum_{j=1}^{t-1} \phi^j (u_{t-j} - u_{t-j-1}), \end{aligned}$$

where the last equality follows if we assume that $r_0 = u_0 = 0$. The last expression in the previous equation may suggest that the quantile series may behave like a random walk.

In this paper, we shall illustrate our results and examples via value at risk (VaR), a quantile estimate that has been frequently used as an effective risk measure in the financial market. Recall that VaR is defined as the worst loss over a target

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horizon such that there is a low, pre-specified probability that the actual loss will be larger (see Jorion, 2006). Mathematically, if we let R denote the next period return and $c \in (0, 1)$ the confidence level, we can write

$$\Pr(R < \text{VaR}) \leq 1 - c.$$

Since its introduction in 1996 (Morgan, 1996), VaR has become the standard measure of market risk. Along this direction, Engle and Manganelli (2004) directly model the quantile and introduce a new class of CAViAR models that specify the evolution of quantile over time based on an autoregressive process; De Rossi and Harvey (2009) propose a state space model for signal (quantiles and expectiles) extraction in an attempt to answer the question of which particular form of non-linear functions should be adopted under the Engle and Manganelli (2004)'s framework. Indeed, it is of both practical and theoretical interest to study if a series of VaRs behave like a random walk. In various literature, such an autoregressive framework of VaRs series is usually imposed. Besides, the results also bears a financial consequence: If the unit root behavior were true, then the best prediction for the next period quantile would be the current value.

We shall consider the following model on $\{\text{VaR}_t\}_{t \geq 0}$, a sequence of VaR's evaluated at time t:

$$\text{VaR}_t = \alpha + \beta \text{VaR}_{t-1} + \epsilon_t, \tag{3}$$

where ϵ_t's are i.i.d. noises. The parameter of interest is the regression coefficient β. We attempt to answer the question of whether or not $\beta = 1$ for two of the commonly used VaR models, namely historical VaR and parametric VaR. In particular, we find that under the efficient market hypothesis setting, i.e. the log stock prices follow a random walk model which is usually captured by a Lévy process (Schoutens, 2003), the unit root feature demonstrated by the VaR using historical simulation approach can be spurious. Parametric VaRs using GARCH(1, 1) models, on the contrary, will not exhibit such a feature.

The rest of the paper is structured as follows: Section 2 introduces the historical VaR and its possible artifact. Section 3 discusses the estimation of VaR following RiskMetrics approach with GARCH(1, 1) model, followed by Section 4 which concludes the article.

2. Historical VaR

Historical simulation in VaR analysis is a procedure for predicting the required quantile by constructing the cumulative distribution function (CDF) of portfolio returns over time. Different from its parametric counterparts, historical simulation does not assume a particular distribution of the asset returns. It is popular amongst practitioners and is accepted by regulators because it is relatively easy to implement. Pérignon and Smith (2010) survey the VaR disclosures of a cross section of 60 US, Canadian and large international banks between 1996 and 2005. Their results conclude that 73% of banks that disclosed their VaR methodology used the historical simulation method.

To compute a VaR series using historical simulation, one has to consider the order statistics of the return series observed under a moving window with a fixed width, say n days. The corresponding non-parametric VaR is then defined as the value of the order statistics below which only $\alpha \in (0, 1)$ percent of the data within the window have smaller values, denoted as $\widehat{\text{VaR}}_t^{(n)}$; for details, readers may refer to Chan and Wong (2015). Mathematically, given a series of returns $\{r_t\}_{t \geq 1}$, we define

$$\widehat{\text{VaR}}_t^{(n)} = \min{}^{(\lfloor (1-\alpha)n \rfloor)} \{r_{t-n+1}, r_{t-n+2}, \ldots, r_t\},$$

where $\lfloor x \rfloor$ and $\min^{(k)} \{X_1, X_2, \ldots, X_n\}$ denote the largest integer not greater than x and the kth-smallest value amongst the n data points, respectively.

Since VaR_t's are not directly observable from the asset price/return series, by adopting the historical simulation approach, one can replace VaR_t in (3) by its historical simulated counterparts $\widehat{\text{VaR}}_t^{(n)}$ as shown below:

$$\widehat{\text{VaR}}_t^{(n)} = \beta_0 + \beta_1 \widehat{\text{VaR}}_{t-1}^{(n)} + e_t. \tag{4}$$

Note that although, under the i.i.d. assumption which is implicitly assumed in the historical simulation procedure (see Hendricks, 1996 and Kuester et al., 2006), the population quantile VaR_t should be a constant, since they are unobservable, their empirical version should conceptually behave like a random walk around the true quantile value. We took a sample of 1511 daily prices for International Business Machines (IBM), Exxon Mobil Corporation (XOM), the Standard and Poor's 500 Index (S&P500) and Hangseng Index (HSI) from 2nd January 2008 to 31st December 2013. The corresponding daily returns were computed as the difference of the log of the prices. The width chosen is 300 days so that the third lowest return within the window corresponds to the historical 99%-VaR estimate. All the numerical results are presented in Table 1, which summarises the value of the estimated parameters, standard error estimates (in brackets), the adjusted R-square statistics and the p-values of the associated augmented Dickey–Fuller test. Our numerical experience suggests that the choice of the window size and/or individual assets does not significantly affect the following conclusion.

It is very tempting to conclude that the historical VaR series follows a random walk model. Standard unit root tests including Dickey–Fuller test (Dickey and Fuller, 1979; Said and Dickey, 1984) does not reject the hypothesis that $\beta_{10} = 1$ as specified in (4), where β_{10} denotes the true value of the regression parameter β_1. The following derivation illustrates how the construction of historical VaR can lead to such spurious random walk model. We use