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Large and moderate deviations for a renewal randomly indexed branching process

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ABSTRACT

The large and moderate deviations for a renewal randomly indexed branching process $\{Z_{N_t}\}$ are derived, where $\{Z_n\}$ is a Galton–Watson process and $\{N_t\}$ is a renewal process which is independent of $\{Z_n\}$.

1. Introduction

Let $\{Z_n, n = 0, 1, 2, \ldots\}$ and $\{N_t, t \geq 0\}$ be two independent stochastic processes on the same probability space $(\Omega, \mathcal{F}, P)$ with the following characters (I) and (II).

(I) $\{Z_n\}$ is a Galton–Watson branching process with an offspring distribution $\{p_i, i = 0, 1, 2, \ldots\}$. Throughout the paper, we assume that our branching process starts from one ancestor, i.e., $Z_0 = 1$ *a.s.*, and $0 < m = \sum_{n=1}^{\infty} np_n < \infty$.

(II) $\{N_t\}$ is a renewal process with the interarrival time X satisfies that $\mu = E(X) \in (0, +\infty)$ and $\sigma^2 = \mathrm{Var}(X) \in (0, +\infty)$.

The continuous time process $\{Z_{N_t}, t \geq 0\}$ is called a renewal randomly indexed branching process (RRIBP). RRIBP is said to be Shröder case (Böttcher case), if $p_0 + p_1 > 0$ $(p_0 + p_1 = 0)$. Similar to the Galton–Watson process, we say that RRIBP is supercritical (critical, subcritical), if $m > 1$ $(m = 1,\ m < 1)$.

RRIBP was introduced by Epps (1996) to study the evolution of stock prices. They consider the case when the indexing process is a Poisson process (denoted by PRIBP). PRIBP can be considered as a homogeneous continuous time Markov chain. Dion and Epps (1999) pointed out that PRIBP is a special class of branching process in random environment and obtained the statistical investigation on various estimates and some parameters.

Wu (2012) showed that if the PRIBP is supercritical and belongs to the Shröder case, the decay rates of the probabilities

$$P\left(\frac{Z_{N_t+1}}{Z_{N_t}} - m \geq x\right) \quad \text{and} \quad P\left(\frac{Z_{N_t+1}}{Z_{N_t}} - m \leq -x\right) \tag{1}$$

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are exponential, where $x > 0$. Similar to Theorem 3 of Athreya (1994), one can show that when the PRIBP belongs to the Böttcher case, the convergence rate of (1) is supergeometric.

On the other hand, Gao and Zhang (2016) showed that $(\lambda t)^{-1}\log Z_{N_t}$ is an estimator of $\log m$ and derived the consistency, asymptotic normality, large deviation and moderate deviation of the estimator when the PRIBP belongs to the Böttcher case. The large deviations of the estimator when PRIBP belongs to the Shröder case were given in Gao and Wang (2015), where the rate function $I(x)$ is different from the Böttcher case for small positive x.

For general RRIBP, Mitov et al. (2010, 2009) considered the critical case. They investigated the probability of non-extinction, the asymptotic behavior of the moments, and also limiting distributions under normalization. Results on subcritical case were done in Mitov and Mitov (2011).

For a special RRIBP, Gao and Zhang (2015) derived similar results as in Gao and Zhang (2016), where the interarrival time satisfies the Erlang(2) distribution. In this paper, we begin with the large deviation principle for a general supercritical RRIBP. In the supercritical case, there is no loss of generality in assuming that the extinction probability $q = 0$, that is, $p_0 = 0$.

Theorem 1.1 (*Large Deviation*). *Assume that $p_0 = 0$, $m > 1$, $E(Z_1^a) < \infty$ for all $a \geq 1$. Let X be the interarrival time. Assume that there exists a constant $t_0 > 0$ such that $M(t_0) := E(\exp(t_0X)) < \infty$. Then for any measurable subset B of $\mathbb{R}$,*

$$
\begin{aligned}
-\inf_{x\in B^o}\psi(x) &\leq \liminf_{t\to\infty}\frac{1}{t}\log P\left(\frac{\log Z_{N_t}}{t}\in B\right)\\
&\leq \limsup_{t\to\infty}\frac{1}{t}\log P\left(\frac{\log Z_{N_t}}{t}\in B\right)\leq -\inf_{x\in\bar{B}}\psi(x),
\end{aligned}
$$

where B^o denotes the interior of B, $\bar{B}$ its closure and

$$
\psi(x) = \begin{cases} \sup_{\theta\in\mathbb{R}}\{x\theta + M^{-1}(m^{-\theta})\}, & x \geq \mu^{-1}p_1\log m;\\ x\log p_1/\log m + M^{-1}(p_1^{-1}), & 0 \leq x < \mu^{-1}p_1\log m\\ +\infty, & x < 0, \end{cases} \tag{2}
$$

where $M^{-1}(\cdot)$ is the inverse function of $M(t) = E(\exp(tX))$.

Remark 1.1. Theorem 1.1 shows that when the branching process belongs to the Böttcher case, (2) is reduced to

$$
\psi(x) = \begin{cases} \sup_{\theta\in\mathbb{R}}\{x\theta + M^{-1}(m^{-\theta})\}, & x \geq 0;\\ +\infty, & x < 0. \end{cases} \tag{3}
$$

Note that $N_t\log m/t$ satisfies the large deviation principle with speed rate function (3) (see Jiang, 1994). Theorem 1.1 shows that $\log Z_{N_t}/t$ satisfies the same large deviation principle with $N_t\log m/t$ when the RRIBP belongs to the Böttcher case. But large deviations of $\log Z_{N_t}/t$ and $N_t\log m/t$ differ for small $x > 0$ when the RRIBP belongs to the Shröder case.

Now we consider the moderate deviations. Let $\{a_t, t \geq 0\}$ be a family of positive numbers satisfying

$$
\frac{a_t}{t}\to 0 \quad\text{and}\quad \frac{a_t}{\sqrt{t}}\to\infty \quad \text{as } t\to\infty.
$$

Definition 1.1. We say that the interarrival time X is strongly nonlattice, if $M(it) \neq 1$ for all $t \neq 0$ and $\liminf_{|t|\to\infty}|1 - M(it)| > 0$.

As in the case of large deviation principle, based on the Gärtner–Ellis theorem and the moderate deviation principle for renewal process, we have

Theorem 1.2 (*Moderate Deviation*). *Assume that $p_0 = p_1 = 0$, the interarrival time X is strongly nonlattice and there exists a constant $t_0 > 0$ such that $M(t_0) := E(\exp(t_0X)) < \infty$. Then for any measurable subset B of $\mathbb{R}$,*

$$
\begin{aligned}
-\inf_{x\in B^o}\frac{\mu^3x^2}{2\sigma^2} &\leq \liminf_{t\to\infty}\frac{t}{a_t^2}\log P\left(\frac{\log Z_{N_t}/\log m - \mu^{-1}t}{a_t}\in B\right)\\
&\leq \limsup_{t\to\infty}\frac{t}{a_t^2}\log P\left(\frac{\log Z_{N_t}/\log m - \mu^{-1}t}{a_t}\in B\right)\\
&\leq -\inf_{x\in\bar{B}}\frac{\mu^3x^2}{2\sigma^2}.
\end{aligned}
$$

The last result shows that the rate function should grow faster than t for the critical and subcritical RRIBP, as in the case of a PRIBP.