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# Fractional spherical random fields

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ABSTRACT

In this paper we study the solutions of different forms of fractional equations on the unit sphere $\mathbf{S}^2_1 \subset \mathbf{R}^3$ possessing the structure of time-dependent random fields. We study the correlation structures of the random fields emerging in the analysis of the solutions of two kinds of fractional equations displaying (Theorem 1) a long-range behaviour and (Theorem 2) a short-range behaviour.



## 1. Introduction

We consider a Brownian motion on the sphere, say $B : t \mapsto \mathbf{S}^2_1$ and a real-valued random field $T : \mathbf{S}^2_1 \mapsto \mathbf{R}$. We study here the time-space dependent random field $X = T \circ B$. We restrict ourselves to Gaussian isotropic random fields $T$ for which the expansion in terms of spherical harmonics holds (see Marinucci and Peccati, 2011 and the references therein). The explicit law of the Brownian motion on $\mathbf{S}^2_1$ was first obtained in Yosida (1949). For Brownian motion processes on $\mathbf{S}^d_1$, (see Karlin and Taylor, 1975, pag. 338).

Time-dependent random fields on the line or on arbitrary Euclidean spaces have been studied by several authors (see, for example, Kelbert et al., 2005; Angulo et al., 2008; Leonenko et al., 2011 and the references therein).

We study here the time-space dependent random fields on the sphere $\mathbf{S}^2_1$, governed by different stochastic differential equations.

We first study random fields emerging as solutions to the Cauchy problem

$$\begin{cases} \left(\gamma - \mathbb{D}_M + \dfrac{\partial^\beta}{\partial t^\beta}\right) X_t(x) = 0, \quad x \in \mathbf{S}^2_1,\ t > 0,\ 0 < \beta < 1,\ \gamma > 0 \\ X_0(x) = T(x), \end{cases} \tag{1.1}$$

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where $\mathbb{D}_M$ is a suitable differential operator defined below and $\frac{\partial^\beta}{\partial t^\beta}$ is the Dzerbayshan–Caputo fractional derivative. We are able to obtain the solution $X_t(x)$ of (1.1) and to show that its covariance function displays a long-memory behaviour.

We then consider the non-homogeneous fractional equation

$$(\gamma - \mathbb{D}_M)^\beta X(x) = T(x), \quad x \in \mathbf{S}_1^2,\ 0 < \beta < 1 \tag{1.2}$$

of which

$$\left(\gamma - \mathbb{D}_M - \varphi \frac{\partial}{\partial t}\right)^\beta X_t(x) = T_t(x), \quad x \in \mathbf{S}_1^2,\ t > 0,\ 0 < \beta < 1,\ \gamma > 0,\ \varphi \geq 0 \tag{1.3}$$

is the time-dependent extension. We obtain a solution to (1.3) which is a random field on the sphere with covariance function with a short-range dependence.

The couple $(B_t, T(x + B_t))$ describes a random motion on the unit-radius sphere with dynamics governed by fractional stochastic equations (1.1) and (1.3).

Random fields similar to those examined here are considered in the analysis of the cosmic microwave background radiation (CMB radiation). In this case, the correlation structure turns out to be very important as well as the angular power spectrum. The angular power spectrum plays a key role in the study of the corresponding random field. In particular, the high-frequency behaviour of the angular power spectrum is related to some anisotropies of the CMB radiation (see for example D'Ovidio, 2014; Marinucci and Peccati, 2011). Such relations have been also investigated in D'Ovidio and Nane (2014) where a coordinates change driven by a fractional equation has been considered.

The time-varying random fields studied here can be useful because the CMB radiation can be affected by some anomalies captured by the angular power spectrum. For a fixed $t$ ($t = 1$ for instance), our models well describe different forms of the angular power spectrum. For $t > 0$, our evolution models well outline anomalies due to instrumental errors, for instance.

Diffusions on the sphere arise in several contexts. At the cellular level, diffusion is an important mode of transport of substances. The cell wall is a lipid membrane and biological substances like lipids and proteins diffuse on it. In general, biological membranes are curved surfaces. Spherical diffusions also crop up in the swimming of bacteria, surface smoothening in computer graphics (Bulow, 2004) and global migration patterns of marine mammals (Brillinger and and Stewart, 1998).

## 2. Preliminaries

### *2.1. Isotropic random fields on the unit-radius sphere*

We introduce notations and some properties of isotropic random fields on the sphere (for a complete presentation see the book by Marinucci and Peccati (2011)). We consider the square integrable isotropic Gaussian random field

$$\{T(x);\ x \in \mathbf{S}_1^2\} \tag{2.1}$$

on the sphere $\mathbf{S}_1^2 = \{x \in \mathbf{R}^3 : |x| = 1\}$ for which

$$\begin{aligned} \mathbb{E}T(gx) &= \mathbb{E}T(x) = 0, \\ \mathbb{E}T^2(gx) &= \mathbb{E}T^2(x) = const \\ \mathbb{E}[T(gx_1)\,T(gx_2)] &= \mathbb{E}[T(x_1)\,T(x_2)] \quad \text{for arbitrary } x_1, x_2 \in \mathbf{S}_1^2, \end{aligned}$$

for all $g \in SO(3)$ where $SO(3)$ is the special group of rotations in $\mathbf{R}^3$. We will consider the spectral representation

$$T(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} a_{l,m} \mathcal{Y}_{l,m}(x) = \sum_{l=0}^{\infty} T^l(x) \tag{2.2}$$

where

$$a_{l,m} = \int_{\mathbf{S}^2} T(x) \mathcal{Y}^*_{l,m}(x) \lambda(dx), \quad -l \leq m \leq l,\ l \geq 0 \tag{2.3}$$

are the Fourier random coefficients of $T$ and $\mathcal{Y}_{l,m}(x) = (-1)^m \mathcal{Y}^*_{l,-m}(x)$. We denote by $\mathcal{Y}^*_{l,m}(x)$ the conjugate of $\mathcal{Y}_{l,m}(x)$. The spherical harmonics $\mathcal{Y}_{l,m}(\vartheta, \varphi)$ are defined as

$$\mathcal{Y}_{l,m}(\vartheta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} Q_{l,m}(\cos\vartheta) e^{im\varphi}, \quad 0 \leq \vartheta \leq \pi,\ 0 \leq \varphi \leq 2\pi$$

for $l \geq 0$ and $|m| \leq l$ where, for $m \geq 0$,

$$Q_{l,m}(z) = (-1)^m (1 - z^2)^{\frac{m}{2}} \frac{d^m}{dz^m} Q_l(z), \quad |z| < 1,\ |m| \leq l,\ \text{and } Q_{l,m}(z) = 0,\ m > l$$