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Statistics & Probability Letters 76 (2006) 1522–1528
STATISTICS & PROBABILITY LETTERS
www.elsevier.com/locate/stapro

On the monotonicity of a function related to the local time of a symmetric Lévy process

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Received 8 August 2005; received in revised form 22 February 2006; accepted 1 March 2006
Available online 18 April 2006

Abstract

Let ψ be the characteristic exponent of a symmetric Lévy process X. The function

$$h(x)=\frac{2}{\pi}\int_0^\infty \frac{1-\cos(\lambda x)}{\psi(\lambda)}\,\mathrm{d}\lambda$$

appears in various studies on the local time of X. We study monotonicity properties of the function h. In case when X is a subordinate Brownian motion, we show that $x\mapsto h(\sqrt{x})$ is a Bernstein function.

MSC: Primary 60J25; secondary 60J55

Keywords: Lévy processes; Subordinate Brownian motion; Local time; Bernstein function

1. Introduction

Let X be a symmetric Lévy process in $\mathbb{R}$ with the characteristic exponent ψ, i.e.,

$$\mathbb{E}\mathrm{e}^{\mathrm{i}\lambda X_t}=\mathrm{e}^{-t\psi(\lambda)}.$$

Throughout this paper we assume that the point 0 is regular for itself, and that the characteristic exponent ψ satisfies

$$\int_0^\infty \frac{1}{1+\psi(\lambda)}<\infty. \tag{1}$$

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[1]The research of this author is supported in part by a joint US–Croatia Grant INT 0302167.
[2]The research of this author is supported in part by MZOS Grant 0037107 of the Republic of Croatia and in part by a joint US–Croatia Grant INT 0302167.

0167-7152/$ - see front matter
doi:10.1016/j.spl.2006.03.013

These two conditions guarantee that the process X admits a local time $L(0,t)$ at zero. Let $T_x = \inf\{s>0 : X_s = x\}$ be the hitting time to $x \in \mathbb{R}$, and let

$$h(x) := \mathbb{E}(L(0, T_x)).$$

Then by Lemma 11 in Chapter 5 of Bertoin (1996)

$$h(x) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\lambda x)}{\psi(\lambda)} \, d\lambda. \tag{2}$$

This function appears often in studies of the local time of Lévy processes. For instance, a monotone rearrangement of this function was used in Barlow (1988) to formulate necessary and sufficient conditions for the joint continuity of the local time. In his study on the most visited sites of X (Marcus, 2001), M.B. Marcus assumed that the function h is strictly increasing on $[0,\infty)$. This assumption on h is not always easy to check. In Section 5 of Marcus (2001), Marcus showed that the so-called stable mixtures satisfy the assumption.

The purpose of this note is to better understand the monotonicity of the function h, and to provide more examples of strictly increasing h. We also show that for subordinate Brownian motions, $x \mapsto h(\sqrt{x})$ is, in fact, a Bernstein function. Under a reasonable additional assumption, it is even a complete Bernstein function.

We start with a simple sufficient condition for h to be increasing. To this end we first rewrite the function h in a different way. It follows from Theorems 16 and 19 in Chapter 2 of Bertoin (1996) that under the assumptions stated in the first paragraph, the q-potential measure U^q of X has a density u^q which is bounded and continuous. From the proof of Lemma 11 in Chapter 5 of Bertoin (1996) we see that h defined by (2) may be written as

$$h(x) = 2 \lim_{q \downarrow 0} (u^q(0) - u^q(x)), \quad x \in \mathbb{R}. \tag{3}$$

Thus if we know that for any $q>0$, the function u^q is decreasing in $[0,\infty)$, then the equation above immediately gives us that h is increasing in $[0,\infty)$. Using this fact and Theorem 54.2 of Sato (1999) we immediately get the following:

Proposition 1.1. *If the Lévy measure ν of the process X is given by*

$$\nu(dx) = n(x)\, dx,$$

for some even function n which is decreasing on $(0,\infty)$, then h is increasing on $[0,\infty)$.

Proof. It follows from Theorem 54.2 of Sato (1999) that when the Lévy measure ν of the process X is given by

$$\nu(dx) = n(x)\, dx$$

for some even function n which is decreasing on $(0,\infty)$, the distribution of X_t is unimodal with mode 0 for every $t>0$. This implies that, for any $q>0$, u^q is a decreasing function on $[0,\infty)$. Therefore, h is increasing on $[0,\infty)$. □

2. Subordinate Brownian motion

In this section we first make a comment that a subordinate Brownian motion satisfies the condition of Proposition 1.1, and then prove that a much stronger result than Proposition 1.1 holds in this case. Let us begin by recalling relevant definitions.

Let $T = (T_t : t \geq 0)$ be a subordinator with Laplace exponent f, that is,

$$\mathbb{E} e^{-\lambda T_t} = e^{-t f(\lambda)},$$

and let $B = (B_t : t \geq 0)$ be a Brownian motion with generator d^2/dx^2. If B and T are independent, then the process $X_t := B(T_t)$ is called a subordinate Brownian motion with subordinator T. It is well known (see, for instance, p. 197 of Sato, 1999) that the characteristic exponent of this subordinate Brownian motion satisfies $\psi(\lambda) = f(\lambda^2)$, that is,

$$\mathbb{E} e^{i\lambda X_t} = e^{-t f(\lambda^2)}.$$