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Parametric bootstrap simultaneous confidence intervals for differences of means from several two-parameter exponential distributions

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ABSTRACT

A parametric bootstrap method is proposed for constructing simultaneous confidence intervals (SCIs) for all pairwise differences of means from several two-parameter exponential distributions. The proposed SCIs are shown to have correct coverage probability asymptotically. Simulation studies show that, comparing with some existing methods, the proposed SCIs are generally closer to the nominal level and possess smaller volumes.

1. Introduction

The probability density function of the two-parameter exponential distribution $\text{Exp}(\mu, \theta)$ is defined as

$$f(x; \mu, \theta) = \frac{1}{\theta}\exp\left(-\frac{x-\mu}{\theta}\right) I_{[\mu,\infty)}(x),$$

where μ is the location parameter, θ is the scale parameter, and $I_A(.)$ is the indicator function of A. The two-parameter exponential distribution family is widely used in the mechanical reliability, life testing, insurance and actuarial science fields, among others. Roy and Mathew (2005) proposed a method based on the concept of generalized confidence intervals to find a generalized confidence limit for the reliability function $e^{-\frac{(x-\mu)}{\theta}}$. Li and Zhang (2010) considered the problem of construct asymptotic confidence interval for the ratio of means of two two-parameter exponential distributions. Kharrati-Kopaei et al. (2013) consider simultaneous fiducial generalized confidence intervals for differences of the location parameters of several exponential distributions under heteroscedasticity. In the quality control study and the experimental design, a more important parameter of interest is the mean lifespan of certain products. For example, it is known that the product quality directly affects the competitive advantage of an enterprise in the market. The quality of the product and its lifespan are

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closely linked. If we assume that several component's life of a mechanical system are all follow life distribution, it is necessary for us to compare the mean life of these parts, timing to replacement and maintenance of these components to ensure the reliability of the product; In experimental design we often consider comparing the life of one or more reference products or one of more test products. Therefore, all pairwise differences of mean life of two or three products have become the urgent problem to address. It is typically assumed that the product life follows a two-parameter exponential distribution $\text{Exp}(\mu, \theta)$, thus its mean life is $\delta = \mu + \theta$, and the question of interest is to compare the differences δ's from several such distributions. Surprisingly, to the best of our knowledge, the literature seems scant in this area. In this paper, we will try to fill this void by constructing simultaneous confidence intervals (SCIs) for differences of two-parameter exponential means using a parametric bootstrap (PB) method. For more about the two-parameter exponential distribution family, see Lawless (1982), Maurya et al. (2011) and the references therein.

The bootstrap approach is a computer method frequently used in applied statistics, which is a type of Monte Carlo method applied on observed data, see Efron and Tibshirani (1993) for more information on this important topic. The bootstrap method can be carried out in either parametric or nonparametric setting. However, the question addressed in this paper is in a strict parametric setting, therefore a parametric bootstrap approach will be constructed accordingly.

The paper is organized as follows. The proposed parametric bootstrap method for constructing SCIs for differences of means of several two-parameter exponential distributions is presented in Section 2, and a theorem on the asymptotic correct coverage probability of the PB SCIs is given in Section 3. In Section 4 simulation results are present to evaluate the empirical coverage probabilities and average volume of the proposed method in comparison to the fiducial generalized simultaneous confidence intervals (FG SCIs) and a nonparametric bootstrap simultaneous confidence intervals (NPB SCIs) procedures. Some concluding remarks are made in Section 5 and the FG SCIs and NPB SCIs methods are briefly described in the Appendix.

2. The parametric bootstrap approach

Let $X_{i1}, \ldots, X_{in_i}$ be random samples from $\text{Exp}(\mu_i, \theta_i)$, $i = 1, \ldots, k$. Suppose that all X_{ij} are independent, $i = 1, \ldots, k$, $j = 1, \ldots, n_i$. Then for $i = 1, \ldots, k$, $\{X_{ij}, j = 1, \ldots, n_i\}$ is an i.i.d. sample from a two-parameter exponential distribution with location parameter μ_i and scale parameter θ_i. The parameter of interest are $\delta_{il} = \delta_i - \delta_l$ for all $i \neq l$, $i, l = 1, \ldots, k$. In the following, we will construct simultaneous confidence intervals for δ_{il} using a parametric bootstrap algorithm.

For any fixed $i = 1, \ldots, k$, let $X_{(1)i}$ be the smallest order statistic of $X_{i1}, \ldots, X_{in_i}$, and $S_i = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (X_{ij} - X_{(1)i})$. $(X_{i(1)}, S_i)$ are complete sufficient statistics for (μ_i, θ_i). Furthermore, it is well know that $X_{(1)i}$ and S_i are independently distributed with

$$X_{(1)i} \sim \text{Exp}\left(\mu_i, \frac{\theta_i}{n_i}\right), \qquad \frac{(2n_i - 2)S_i}{\theta_i} \sim \chi^2(2n_i - 2) \tag{2.1}$$

where $\chi^2(m)$ denotes a central chi-square distribution with degrees of freedom m. As $n_i \to \infty$, it is easy to verify that $X_{i(1)}, S_i$ converge to μ_i, θ_i in probability, respectively. Therefore, a reasonable estimate of δ_i is

$$\widehat{\delta_i} = X_{(1)i} + S_i, \quad i = 1, \ldots, k.$$

Note that

$$\text{Var}(\widehat{\delta_i} - \widehat{\delta_l}) = \frac{\theta_i^2}{n_i^2} + \frac{\theta_i^2}{n_i - 1} + \frac{\theta_l^2}{n_l^2} + \frac{\theta_l^2}{n_l - 1}, \quad i \neq l \tag{2.2}$$

and an unbiased estimator of (2.2) is

$$\text{V}_{il} = \frac{n_i - 1}{n_i^3} S_i^2 + \frac{1}{n_i} S_i^2 + \frac{n_l - 1}{n_l^3} S_l^2 + \frac{1}{n_l} S_l^2. \tag{2.3}$$

Define

$$D_n = \max_{i \neq l} \left| \frac{(\widehat{\delta_i} - \widehat{\delta_l}) - (\delta_i - \delta_l)}{\sqrt{\text{V}_{il}}} \right|. \tag{2.4}$$

Then the approximate $100(1 - \alpha)\%$ two-sided SCIs for $\delta_i - \delta_l (i \neq l)$ can be constructed as

$$(\widehat{\delta_i} - \widehat{\delta_l}) \pm q_\alpha \sqrt{\text{V}_{ij}}, \quad i, \ l = 1, \ldots, k \ (i \neq l) \tag{2.5}$$

where q_α denotes the approximate $(1 - \alpha)$th quantile of the distribution of D_n. Unfortunately, the exact distribution of D_n is very hard to find, if not impossible. This motivates us to find an alternative way for constructing the SCIs for δ_{il} based on the characteristic of D_n.

To begin with, note that the distribution of D_n does not depend on the values of μ_i's, so without loss of generality, all μ_i's are assumed to be zeros. Based on the D_n in (2.4), and the fact (2.1) we can define the PB analogues of D_n as follows. Let

$$X_{(1)i}^{PB} \sim \text{Exp}\left(0, \frac{s_i}{n_i}\right), \qquad S_i^{PB} \sim \frac{s_i \chi^2(2n_i - 2)}{2n_i - 2}$$