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Mean-field backward stochastic differential equations with subdifferential operator and its applications[☆]

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ABSTRACT

In this paper, we deal with a class of mean-field backward stochastic differential equations with subdifferential operator corresponding to a lower semi-continuous convex function. By means of Yosida approximation, the existence and uniqueness of the solution is established. As an application, we give a probability interpretation for the viscosity solutions of a class of nonlocal parabolic variational inequalities.

1. Introduction

The general nonlinear case of backward stochastic differential equations (BSDEs, in short) was first introduced by Pardoux and Peng (1990). Since then, a lot of works have been devoted to the study of the theory of BSDEs as well as their applications. This is due to the connections of BSDEs with mathematical finance, stochastic optimal control as well as stochastic games and partial differential equations (PDEs, in short) (see e.g. El Karoui et al., 1997a, Hamadène and Lepeltier, 1995a,b, Pardoux, 1998, 1999, Pardoux and Peng, 1992, and Peng, 1991, 1993).

Among the BSDEs, El Karoui et al. (1997b) introduced a special class of reflected BSDEs, which is a BSDE but the solution Y is forced to stay above a given lower barrier. By means of these kinds of BSDEs, they provided a probabilistic formula for the viscosity solution of an obstacle problem for a class of parabolic PDEs. Pardoux and Răşcanu (1998) (and Pardoux and Răşcanu, 1999 for the framework of Hilbert space) studied the existence and uniqueness of the solutions of BSDEs, on a random (possibly infinite) time interval, involving a subdifferential operator (which are also called backward stochastic

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variational inequalities) in order to give a probabilistic interpretation for the viscosity solution of some parabolic and elliptic variational inequalities. Its extension to the probabilistic interpretation of the viscosity solution of the parabolic variational inequality (PVI, in short) with a mixed nonlinear multivalued Neumann–Dirichlet boundary condition was recently given in Maticiuc and Răşcanu (2010).

Mathematical mean-field approaches have been used in many fields, not only in physics and Chemistry, but also recently in economics, finance and game theory, see for example, Lasry and Lions (2007) and the references therein.

On the other hand, *McKean–Vlasov* stochastic differential equation of the form

$$dX_t = b(X_t, \mu_t)dt + dW_t, \quad t \in [0, T], \qquad X_0 = x,$$

where

$$b(X_t, \mu_t) = \int_\Omega b(X_t(\omega), X_t(\omega'))P(d\omega') \equiv \int_R b(\xi, y)\mu_t(dy)|_{\xi=X_t} \equiv E[b(\xi, X_t)]|_{\xi=X_t},$$

$b : R^n \times R \to R$ being a (locally) bounded Borel measurable function and $\mu_t(\cdot)$ being the probability distribution of the unknown process X_t, was suggested by Kac (1956) as a stochastic toy model for the Vlasov kinetic equation of plasma and the study of which was initiated by McKean (1966). Since then, many authors made contributions on McKean–Vlasov type SDEs and their applications, see for example, Ahmed (2007), Ahmed and Ding (1995), Borkar and Kumar (2010), Chan (1994), Crisan and Xiong (2010), Kotelenez (1995), Kotelenez and Kurtz (2010) and the references therein.

Recently, Buckdahn et al. (2009a) introduced a new kind of BSDEs called as mean-field BSDEs. Furthermore, Buckdahn et al. (2009b) deepened the investigation of mean-field BSDEs in a rather general setting. They proved the existence and uniqueness as well as a comparison principle of the solutions for mean-field BSDEs. Moreover, they established the existence and uniqueness of the viscosity solution for a class of nonlocal PDEs with the help of the mean-field BSDE and a McKean–Vlasov forward equation.

Since the works of Buckdahn et al. (2009a,b) on the mean-field BSDEs, there are many efforts devoted to its generalization. Shi et al. (2011) introduced and studied mean-field backward stochastic Volterra integral equations. Xu (2012) obtained the existence and uniqueness of solutions for mean-field backward doubly stochastic differential equations, and gave the probabilistic representation of the solutions for a class of stochastic partial differential equations by virtue of mean-field BDSDEs. Li (2014) studied reflected mean-field BSDEs in a purely probabilistic method, and gave a probabilistic interpretation of the obstacle problems of the nonlinear and nonlocal PDEs by means of the reflected mean-field BSDEs.

Motivated by the above works, the present paper aims to deal with a class of mean-field BSDEs with subdifferential operator corresponding to a lower semi-continuous convex function with the form

$$\begin{cases} -dY_t + \partial\varphi(Y_t)dt \ni E'[f(t, Y'_t, Z'_t, Y_t, Z_t)]dt - Z_t dW_t, \quad 0 \le t \le T, \\ Y_T = \xi, \end{cases} \tag{1.1}$$

where $\partial\varphi$ is the subdifferential operator of a proper, convex and lower semicontinuous function φ, ξ is called as the terminal condition. More details refer to Section 2.

The first goal of this paper is to find a triple of adapted processes (Y, Z, U) in an appropriate space such that mean-field BSDE (1.1) hold (see Definition 2.1). Then, it allows us to establish the unique viscosity solution of the following nonlocal parabolic variational inequality

$$\begin{cases} \dfrac{\partial u(t, x)}{\partial t} + Au(t, x) + E[f(t, X_t^{0,x_0}, x, u(t, X_t^{0,x_0}), u(t, x), Du(t, x) \cdot E[\sigma(t, X_t^{0,x_0}, x)])] \in \partial\varphi(u(t, x)), \\ u(T, x) = E[h(X_T^{0,x_0}, x)], \quad x \in R^n \end{cases} \tag{1.2}$$

with $Au(t, x) := \frac{1}{2}\mathrm{tr}(E[\sigma(t, X_t^{0,x_0}, x)]E[\sigma(t, X_t^{0,x_0}, x)]^T D^2 u(t, x)) + \langle E[b(t, X_t^{0,x_0}, x)], Du(t, x)\rangle$.

The paper is organized as follows. In Section 2, we introduce some notations, basic assumptions and preliminaries. Section 3 is devoted to the proof of the existence and uniqueness of the solution to the mean-field BSDEs with subdifferential operator. In Section 4, we give a probability interpretation for the viscosity solution of a class of nonlocal parabolic variational inequalities by means of the mean-field BSDEs with subdifferential operator.

2. Notations, preliminaries and basic assumptions

Let $T > 0$ be a fixed deterministic terminal time. Let $\{W_t\}_{t\ge0}$ be a d-dimensional standard Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, P)$. We denote by $\mathbb{F} = \{\mathcal{F}_s, \ 0 \le s \le T\}$ the natural filtration generated by $\{W_t\}_{0\le t\le T}$ and augmented by all P-null sets, i.e., $\mathcal{F}_s = \sigma\{W_r, r \le s\} \vee \mathcal{N}_P, \ s \in [0, T]$, where $\mathcal{N}_P$ is the set of all P-null subsets.

In what follows, we need the following spaces.

- $S^2_{\mathbb{F}}(0, T; R)$: the space of $\mathbb{F}$-adapted processes $Y : \Omega \times [0, T] \to R$ such that $E\left[\sup_{t\in[0,T]} |Y_t|^2\right] < +\infty$;
- $H^2_{\mathbb{F}}(0, T; R^n)$: the space of $\mathbb{F}$-progressively measurable processes $\psi : \Omega \times [0, T] \to R^n$ such that $\|\psi\|^2 := E\int_0^T |\psi_t|^2 dt < +\infty$.