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# The Fourier dimension of Brownian limsup fractals

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ABSTRACT

Robert Kaufman's proof that the set of rapid points of Brownian motion has a Fourier dimension equal to its Hausdorff dimension was first published in 1974. Some of the original arguments presented by Kaufman are not clear, hence this paper presents a new version of the construction and incorporates some recent results in order to establish a slightly weaker version of Kaufman's theorem. This weaker version still implies that the Fourier and Hausdorff dimensions of Brownian rapid points are equal. The method of proof can then be extended to show that functionally determined rapid points of Brownian motion also form Salem sets for absolutely continuous functions of finite energy.



## 1. Introduction

Brownian motion has proved itself a rich source of sets with interesting dimensional properties. It is well known that the level sets have Hausdorff dimension 1/2, almost surely, and Kahane showed that they have equal Fourier dimension (Kahane, 1983), implying that they form Salem sets. The corresponding question for fractional Brownian motion remains open, although preliminary results can be found in Fouché and Mukeru (2013). Although Salem sets arise commonly in the images of Brownian motion, the graph of Brownian motion almost surely does not form a Salem set, as shown in Fraser et al. (2014), based on a result in Orponen (2014).

It has been accepted since 1974 that the rapid points of Brownian motion also form Salem sets, due to the work of Orey and Taylor (1974) and Kaufman (1974). A study of Kaufman's paper on the Fourier dimension of the set of rapid points has, however, raised several questions regarding the proof. Due to the significance of the result to the study of sample path properties of Brownian motion, it was felt that an updated and expanded version of the proof was merited. Furthermore, such methods can shed light on random fractals whose Hausdorff dimensions have been found, but whose Fourier dimensions are not yet known.

As in Potgieter (2012), we take the following as the definition of one dimensional Brownian motion:

**Definition 1.1.** Given a probability space $(\Omega, \mathcal{B}, \mathbf{P})$, a *Brownian motion* is a stochastic process $X$ from $\Omega \times [0, 1]$ to $\mathbb{R}$ satisfying the following properties:

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1. Each path $X(\omega,\cdot):[0,1]\to\mathbb{R}$ is almost surely continuous.
2. $X(\omega,0)=0$ almost surely.
3. For $0\le t_1<t_2\cdots<t_n\le 1$, the random variables $X(\omega,t_1), X(\omega,t_2)-X(\omega,t_1),\dots,X(\omega,t_n)-X(\omega,t_{n-1})$ are independent and normally distributed with mean 0 and variances $t_1, t_2-t_1,\dots,t_n-t_{n-1}$, respectively.

We let $X$ denote a continuous Brownian motion on [0, 1], and let $X(t)$ denote the value of a sample path $X(\omega)$ at $t$, if there is no confusion as to which sample path is used. We define the random set of rapid points relative to the parameter $\alpha$ (also referred to as the $\alpha$-rapid points) by

$$E_\alpha(\omega)=\left\{t\in[0,1]:\limsup_{h\to 0}\frac{|X(t+h)-X(t)|}{\sqrt{2|h|\log 1/|h|}}\ge\alpha\right\}. \tag{1.1}$$

Again, we discard $\omega$ and just write $E_\alpha$ when there is no confusion as to the sample path. The sets $E_\alpha$ have Lebesgue measure 0 almost surely. They are exceptional points of rapid growth, since the usual local growth behaviour is described by Khintchine's law of the iterated logarithm (Khintchine, 1923):

$$\mathbf{P}\left\{\limsup_{h\to 0}\frac{X(t_0+h)-X(t_0)}{\sqrt{2|h|\log\log 1/|h|}}=1\right\}=1, \tag{1.2}$$

for any prescribed $t_0$.

Orey and Taylor (1974) showed that the sets $E_\alpha$ have Hausdorff dimension $1-\alpha^2$. An elementary proof of the result can be found in Potgieter (2012).

In this paper we define the Fourier–Stieltjes transform of a measure $\mu$ with support $A\subseteq\mathbb{R}$ as

$$\hat{\mu}(u)=\int_A e^{ixu}d\mu(x). \tag{1.3}$$

We have the following definition:

**Definition 1.2.** An $M_\beta$-set is a compact set in $\mathbb{R}^n$ which carries a measure $\mu$ such that $|\hat{\mu}(u)|=o(|u|^{-\beta})$ as $|u|\to\infty$, for some $\beta>0$. For a compact set $E$, we call the supremum of the $\alpha$ such that $E$ is an $M_{\alpha/2}$-set, the *Fourier dimension* of $E$. We shall denote this by $\dim_F E$.

The Fourier dimension is a somewhat more elusive character than Hausdorff dimension. They are usually different, as for instance in the case of the triadic Cantor set, which has strictly positive Hausdorff dimension but a Fourier dimension of 0 (Kechris and Louveau, 1987). Indeed, it can be shown that Hausdorff dimension majorises Fourier dimension. When the dimensions coincide, the set is called a Salem set. Kaufman's paper claims to establish the following result:

**Theorem 1.1** (*Kaufman, 1974*)**.** *With probability* 1*, a certain compact subset of* $E_\alpha$*, the* $\alpha$*-rapid points of a given Brownian motion* $X$*, carries a probability measure* $\mu$ *such that* $|\hat{\mu}(\xi)|=o(|\xi|^{\frac{1}{2}(\alpha^2-1)})$.

We have not been able to replicate the full strength of Kaufman's theorem, since we can only establish the decay in Theorem 1.1 for values smaller than $1-\alpha^2$. Whether it will hold for $1-\alpha^2$ itself is unknown. However, this weaker result still gives the required Fourier dimension:

**Theorem 1.2.** *Given* $\alpha$, $0<\alpha<1$*, then for each* $\gamma<1-\alpha^2$ *there exists, with probability* 1*, a measure* $\mu$ *on a compact subset of* $E_\alpha$ *such that* $|\hat{\mu}(u)|=o(|u|^{-\frac{\gamma}{2}})$*. That is,* $E_\alpha$ *is a set of Fourier dimension* $1-\alpha^2$.

(The fact that Hausdorff dimension majorises Fourier dimension guarantees that the Fourier dimension of $E_\alpha$ cannot be larger than $1-\alpha^2$.)

The structure of the proof is initially similar to Kaufman's. Specifically, we use two of the same lemmas that Kaufman did, albeit slightly altered. These do not require innovative techniques to prove, only a refinement of Kaufman's original arguments. We prove the first of these in Section 2. The second lemma, the domain of which had to be slightly limited, is proved in Section 3. This is followed by the proof of the main theorem, where we need to depart from Kaufman's construction to a larger extent. The measures constructed in the various stages in his paper do not necessarily converge to a probability measure, and it is only inferred that the limit measure has the requisite properties, never proven. It has proved difficult to mend the construction using only the Fourier-analytical methods originally considered. It became necessary to introduce certain dimensional arguments (Potgieter, 2012) which give us estimates on the actual number of intervals considered at each stage. In this way, it is quite obvious that the Hausdorff dimension of the set is instrumental in guaranteeing the desired Fourier dimension, and in such a precise way that the set becomes Salem. In Section 4, we define functionally determined rapid points of Brownian motion, which have properties similar to those of the sets $E_\alpha$. The Hausdorff dimension of such sets was found by Deheuvels and Mason (1994), in an extension of the Orey–Taylor theorem. Further interesting extensions were obtained by Khoshnevisan et al. (2000). We extend Kaufman's result to show that the functionally determined rapid points also form Salem sets, almost surely.