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On the empirical process of strongly dependent stable random variables: asymptotic properties, simulation and applications

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ABSTRACT

This paper analyzes the limit properties of the empirical process of α-stable random variables with long range dependence. The α-stable random variables are constructed by non-linear transformations of bivariate sequences of strongly dependent gaussian processes. The approach followed allows an analysis of the empirical process by means of expansions in terms of bivariate Hermite polynomials for the full range $0 < \alpha < 2$. A weak uniform reduction principle is provided and it is shown that the limiting process is gaussian. The results of the paper differ substantially from those available for empirical processes obtained by stable moving averages with long memory. An application to goodness-of-fit testing is discussed.

1. Introduction

Consider a sequence of random variables (rv) $X_1, \ldots, X_n$, with common continuous cumulative distribution function (CDF) F, constituting a sample from a strictly stationary and ergodic time series $\{X_i, i \in \mathbb{Z}\}$ where $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. For $\mathbb{1}\{A\}$ being the indicator function of the event A, let F_n denote the empirical distribution function (EDF) of the sequence, i.e. $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{X_i \le x\}$. It is well known that the empirical process (EP)

$$\sqrt{n}(F_n(x) - F(x)) \tag{1}$$

converges to a non-degenerate gaussian process either in the case where $\{X_i\}$ is a sequence of *i.i.d.* or weakly dependent rv.

The behavior of the EP is quite different in the case of long range dependence (LRD) where proper normalizing constants are of order $n^{D/2}$, $0 < D < 1$ and the weak limit, if it exists, is a degenerate process in x.

This paper studies the weak limit of $(F_n(x) - F(x))$, properly normalized, when the sample is formed by a sequence of strongly dependent stable random variables with index of stability $0 < \alpha < 2$.

One of the mainstream approaches in the study of LRD processes is via expansions, by means of orthogonal polynomials, of non-linear functionals of gaussian LRD processes. In the case discussed here, if F denotes the CDF of a stable rv X and Φ the CDF of a standard normal rv Z, one has $\mathbb{1}\{X \le x\} = \mathbb{1}\{F^{-1} \circ \Phi(Z) \le x\} = \mathbb{1}\{Z \le F \circ \Phi^{-1}(x)\}$; in this framework it is quite simple to provide an expansion of the indicator function in an appropriate L_2 space. However, given that analytic expressions of F^{-1}, with a few exceptions, are not available, this approach may not be optimal if one, for simulation, validation and testing purposes, wishes to generate stable rv given a sequence of LRD gaussian rv.

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In this paper an approach based on a bivariate expansion is proposed. This will allow to provide fast and reliable methods of stable rv generation starting from an LRD gaussian sequence and, at the same time provide an analytic framework for the analysis of the EP. Some key results in this respect are due to Chambers et al. (1976) and Weron (1996) as far as stable rv are concerned. Specific papers considering the EP of non-linear transformation of LRD gaussian sequences discussing techniques relevant here are those of Dehling and Taqqu (1989), Csörgö and Mielniczuk (1996), Leonenko and Sakhno (2001) and Leonenko et al. (2002). We also refer the interested reader to the excellent reviews of Dehling and Philipp (2002) for a general discussion on EP techniques and Koul and Surgailis (2002) for a specific analysis of the LRD case. Other relevant literature discussing bivariate (and multivariate) expansion on non-linear functionals of LRD gaussian sequences and other bivariate expansions are Arcones (1994), Leonenko and Taufer (2001), Lévy-Leduc et al. (2011), Lévy-Leduc and Taqqu (2014).

Another mainstream approach in the study of LRD processes, which will not be discussed here, is based on linear processes (or moving averages). In this line of study, specific papers devoted to the EP are those of Ho and Hsing (1996), Giraitis and Surgailis (1999) and Koul and Surgailis (2001) which, in particular, consider the case of stable innovations with $1 < \alpha < 2$ and where a non-gaussian weak limit is obtained.

It is worth noting that the approach followed here provides a discussion of the full range $0 < \alpha < 2$, new to the literature, and provides a gaussian weak limit.

The results obtained can find applications in the analysis of statistical functionals based on the EP. Relevant and recent examples in the literature concern the analysis of goodness of fit tests, such as, e.g. Taufer (2009), Koul and Surgailis (2010), Dehling et al. (2013), Koul et al. (2013), Ghosh (2013).

This paper is organized as follows: Section 2 contains background arguments while in Section 3 the EP of stable rv is discussed. A final section presents applications and simulations to substantiate the theoretical findings. The paper is accompanied by a *Supplemental material* pdf file which provides and discusses a *Mathematica 9* code to reproduce all the formulae and simulations discussed here; a running *Mathematica 9* notebook version of it is also provided (see Appendix A).

2. Background

In this section, some needed key features of stable rv will be recalled and a bivariate expansion, in terms of Hermite polynomials, of the EP of LRD stable random variables will be provided.

In order to define exactly the sequence X of stable rv we state the following assumption where the classical set-up for a sequence of LRD gaussian random variables is defined:

Assumption 1. Let $Z_i^{(1)}$ and $Z_i^{(2)}$ be independent copies of a sequence of gaussian random variables with null mean and unit variance and, for $j = 1, 2$, $r(k) = \mathrm{E}(Z_i^{(j)}, Z_{i+k}^{(j)}) = L(k)k^{-D}$ with $L(k)$ a slowly varying function and $0 < D < 1$.

2.1. Stable rv

For $0 < \alpha \le 2$, write $X \sim S_\alpha(\beta, \sigma, \mu)$ to denote an α-stable rv with asymmetry $\beta \in [-1, 1]$, scale $\sigma > 0$ and location $\mu \in \mathbb{R}$, with characteristic function ψ given by (here $i = \sqrt{-1}$)

$$\log \psi(z) = \begin{cases} i\mu z - \sigma^\alpha |z|^\alpha \left[1 - i\beta \operatorname{sign}(z) \tan\left(\frac{\pi\alpha}{2}\right)\right], & \alpha \neq 1 \\ i\mu z - \sigma|z| \left[1 + i\beta \operatorname{sign}(z) \frac{2}{\pi} \log(|z|)\right], & \alpha = 1. \end{cases} \tag{2}$$

An alternative representation, justified by considerations of analytic nature (see Zolotarev, 1986, Theorem C.3), which will be relevant for our development is

$$\log \psi(z) = \begin{cases} i\mu z - \sigma_2^\alpha |z|^\alpha \exp\left\{-i\beta_2 \operatorname{sign}(z) \frac{\pi}{2} K(\alpha)\right\}, & \alpha \neq 1 \\ i\mu z - \sigma_2 |z| \left[\frac{\pi}{2} + i\beta_2 \operatorname{sign}(z) \log(|z|)\right], & \alpha = 1 \end{cases} \tag{3}$$

where $K(\alpha) = \alpha - 1 + \operatorname{sign}(1 - \alpha)$. The parameters of representations (2) and (3) can be connected: for $\alpha = 1$, it holds that $\beta_2 = \beta$ and $\sigma_2 = 2\sigma/\pi$; while for $\alpha \neq 1$ one has σ and σ_2, β and β_2 related by the equations

$$\tan\left(\frac{\beta_2 \pi K(\alpha)}{2}\right) = \beta \tan\left(\frac{\pi\alpha}{2}\right), \qquad \sigma_2 = \sigma\left(1 + \beta^2 \tan^2\left(\frac{\pi\alpha}{2}\right)\right)^{1/(2\alpha)}. \tag{4}$$

Chambers et al. (1976) introduced a fast algorithm for generating α-stable rv; later Weron (1996) provided proof details about the algorithm; using when possible, for continuity, the notation established in Weron (1996), define

$$\gamma = \gamma(Z^{(1)}) = \pi\Phi(Z^{(1)}) - \pi/2 \quad \text{and} \quad W = W(Z^{(2)}) = -\log\left(1 - \Phi(Z^{(2)})\right) \tag{5}$$