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Characteristic function of the positive part of a random variable and related results, with applications

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ABSTRACT

Let X be an arbitrary real-valued random variable (r.v.), with the characteristic function (c.f.) f. Integral expressions for the c.f. of the r.v.'s $\max(0, X)$ in terms of f are given, as well as other related results. Applications to stock options and random walks are presented. In particular, a more explicit and compact form of Spitzer's identity is obtained.

1. Introduction

The positive part, $X_+ := 0 \vee X = \max(0, X)$, of a random variable (r.v.) X arises in various contexts, including inequalities and extremal problems in probability, statistics, operations research, and finance; see e.g. Spitzer (1956), Pinelis (1994), Bentkus (2004) and Pinelis (2014a,b) and further references therein. In particular, in finance $(S-K)_+$ is the value of a call option with strike price K when the underlying stock price is S. Also, the absolute value of a r.v. is easily expressible in terms of the positive-part operation: $|X| = X_+ + X_-$, where $X_- := (-X)_+$. Motivated by different needs and using rather different methods, Brown (1970) and Pinelis (2011) provided various expressions for the power moments $\mathsf{E}\,X_+^p$ $(p > 0)$ of the positive part X_+ in terms of the Fourier or Fourier–Laplace transform of (the distribution of) X.

However, expressions for the characteristic function (c.f.) f_{X_+} of the r.v. X_+ in terms of the c.f. f_X of X appear to be absent in the existing literature. Here such expressions are provided, as well as related ones. We also include applications to stock options and random walks. In particular, a more explicit and compact form of Spitzer's identity is obtained.

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2. Basic results

Let X be any real-valued r.v., and let then $f = f_X$ denote the characteristic function (c.f.) of X, so that $f(t) = \mathsf{E}\, e^{itX}$ for all real t. Introduce

$$(J_a f)(t) := \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-iua} f(t+u)\, \frac{du}{u} \tag{1}$$

$$= \frac{1}{4\pi i} \int_{-\infty}^{\infty} \big[e^{-iua} f(t+u) - e^{iua} f(t-u)\big]\, \frac{du}{u}. \tag{2}$$

Here and subsequently, $a, b, s, t, u, x, \alpha, \beta, \gamma$ stand for arbitrary real numbers (unless otherwise specified) and the integral $\int_{-\infty}^{\infty}$ is understood in the principal-value sense, so that

$$(J_a f)(t) = \lim_{\varepsilon\downarrow 0,\, A\uparrow\infty} (J_{a;\varepsilon,A} f)(t), \tag{3}$$

where

$$(J_{a;\varepsilon,A} f)(t) := \frac{1}{2\pi i} \int_{\varepsilon,A} e^{-iua} f(t+u)\, \frac{du}{u} \quad \text{and} \quad \int_{\varepsilon,A} := \int_{\varepsilon}^{A} + \int_{-A}^{-\varepsilon}. \tag{4}$$

The equality in (2) follows by the change of the integration variable $u \mapsto -u$. Because of the singularity of the integrand in (1) at $u = 0$, the expression of $(J_a f)(t)$ in (2) will usually be more convenient in computation than the expression of $(J_a f)(t)$ in (1). By the same change of the integration variable, one has the following parity property of the transformation J_a:

$$(J_a f)(-t) = -(J_{-a} f^-)(t), \tag{5}$$

where $f^-(u) := f(-u)$ for all real u. Of course, if f is a c.f., then $f^- = \bar{f}$, where, as usual, the horizontal bar above the symbol denotes the complex conjugation.

Clearly, one may also write

$$(J_a f)(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} [e^{-iua} f(t+u) - f(t)]\, \frac{du}{u}.$$

Moreover, in place of $\frac{1}{4\pi i} \int_{-\infty}^{\infty}$ in (2) one may write $\frac{1}{2\pi i} \int_0^{\infty}$, where $\int_0^{\infty} := \lim_{\varepsilon\downarrow 0,\, A\uparrow\infty} \int_{\varepsilon}^{A}$.

We shall now see that the integral in (1) always exists — as long as f is a c.f.; moreover, the value of $(J_a f)(t)$ is no greater than $\frac{1}{2}$ in modulus. In fact, one has the following basic proposition.

Proposition 1. *One has*

$$(J_a f)(t) = \tfrac{1}{2}\, \mathsf{E}\, e^{itX} \operatorname{sign}(X - a). \tag{6}$$

Moreover, for all r.v.'s X, all a, and all ε and A such that $0 < \varepsilon < A < \infty$

$$\big|(J_{a;\varepsilon,A} f)(t)\big| < 1. \tag{7}$$

Proof of Proposition 1. Consider first the (extreme) case when the real-valued r.v. X takes only one value, say x. Then $f(t) = e^{itx}$ and hence

$$(J_a f)(t) = \frac{e^{itx}}{2\pi i} \int_{-\infty}^{\infty} e^{iu(x-a)}\, \frac{du}{u} = \frac{e^{itx}}{2\pi} \int_{-\infty}^{\infty} \frac{\sin u(x-a)}{u}\, du = \frac{1}{2}\, e^{itx} \operatorname{sign}(x-a),$$

so that (6) holds in this case. Similarly, still in this same extreme case,

$$2\pi\, |(J_{a;\varepsilon,A} f)(t)| = \left| \int_{\varepsilon,A} \frac{\sin u(x-a)}{u}\, du \right| \leqslant \int_{-\pi}^{\pi} \frac{\sin v}{v}\, dv < \int_{-\pi}^{\pi} dv = 2\pi,$$

so that (7) holds in this case as well. It remains to use Fubini's theorem to obtain (6) and (7) in general. $\square$

Consider the r.v.

$$X_+ := 0 \vee X = \max(0, X),$$

the positive part of the r.v. X. In view of the obvious identity

$$2e^{itX_+} = 1 + e^{itX} + e^{itX} \operatorname{sign} X - \operatorname{sign} X, \tag{8}$$

one immediately obtains the following corollary of Proposition 1, which gives an expression of the c.f. of X_+ in terms of the c.f. of X.