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The degree sequences of an asymmetrical growing network

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ABSTRACT

In this paper, we use utility to describe the attractive effect and then study simple asymmetrical evolving model, considering both preferential attachment and the randomness of the utility. The model is defined so that, at each integer time t, a new vertex, with m edges attached to it, is added to the graph. The new edges added at time t are then preferentially connected to older vertices, i.e., conditionally on $G(t-1)$, the probability that a given edge is connected to vertex i is proportional to its utility at time $t-1$. The main result is that the asymptotical degree sequence for this process is a power law with exponent $2+1/p$.

1. Introduction

Networks are ubiquitous in nature and society, which describe various complex systems, such as the World Wide Web (Albert et al., 1999; Adamic and Huberman, 2000), movie actor network (Barabási and Albert, 1999; Watts and Strogatz, 1998), social networks (Watts, 2003) and so on. Especially, Barabási and Albert (1999) point out that many complex real world networks cannot be adequately described by the classical Erdös–Rényi random graph model (Bollobás, 2001), where the possible edges are included independently, with the same probability p. In E–R model the degree distribution is approximately Poisson with parameter np, while in real life, many networks appear to exhibit the scale-free property. That is, the degree distribution $P(k)$ gives a power-law behavior. The ubiquity of scale-free networks has inspired an extensive study of the topological properties. There are two main aspects that lead to the emergence of the scale-free property, such as growing pattern and preferential attachment. Growing pattern captures the fact that networks are assembled through the addition of new vertices to the networks, while preferential attachment implies that the old vertices with higher degree are more likely to be attached to (Bollobás et al., 2001; Szymański, 2005). Furthermore, the weighted networks have drawn strong attention, in which the equal edge probabilities of the Erdös–Rényi random graph model is replaced by edge occupation statuses that are conditionally independent, given some vertex weights. The weights can be chosen to be random (i.i.d weights (van den Esker et al., 2006; Britton et al., 2005) or fixed (Chung and Lu, 2002a,b, 2006a)).

In most studies, the weight of the link functions symmetrically with respect to its two endpoints, i.e., the weight influences symmetrically the link increment of its two endpoints (van der Hofstad, 2007). However, asymmetrical evolving networks are seldom considered in the previous works. Partly inspired by Zheng et al. (2007), we also introduce utility to describe the asymmetrical property when a connection between two vertices is completed, which may be popular in many real networks. For example, in movie actor network, two actors are connected so as long as they perform the same movie, but their roles, leading actor or not, may be different, so that it is necessary to build a function to describe such preferential connectivity. Similarly, in scientific collaboration networks, there is a link between two authors if they have ever coauthored a work, but their contribution must be different. Because of their different importance, their change of attractive effect of the two endpoints may vary asymmetrically after connection. This paper attempts to give a rigorous mathematical analysis of the improvement version of the model proposed by Zheng et al. (2007): in the process of network growing, at each time

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t, we consider the new vertex to be positive and its utility increased by 1 precisely, when one of its new edges is connected to an old vertex present in the networks, while the chosen vertex is seen to be passive and its utility is increased by 1 with probability p. Our investigation shows that the degree sequence obeys power-law distribution for $0 < p \leq 1$, but it has a generalized geometrical distribution when $p = 0$.

1.1. Definition of the model

The model that we consider is described by a graph process $\{G(t)\}_{t\geq 1}$. To define it, let $G(1)$ be a graph consisting of two vertices v_0 and v_1 with m (a positive integer) edges joining them. For $t \geq 2$, the graph $G(t)$ is constructed from $G(t-1)$ in such a way that a vertex v_t, with associated weight m, is added to the graph $G(t-1)$, and the edge set is updated by adding m edges between the vertex v_t and the vertices $\{v_0, v_1, \ldots, v_{t-1}\}$. Thus, m is the initial degree of vertex v_t, we also let the initial utility of the vertex v_t be its initial degree. Write $d_0(s), \ldots, d_{t-1}(s)$ and $u_0(s), \ldots, u_{t-1}(s)$ respectively for the degrees and utilities of the vertices $\{v_0, v_1, \ldots, v_{t-1}\}$ at time $s \geq t-1$. At time t, when the vertex v_i already present in the network is hit by a new edge emanating from the new added vertex v_t, the utility set is updated as follows: for the old vertex $v_i (0 \leq i \leq t-1)$,

$$u_i(t) = \begin{cases} u_i(t-1) + 1, & \text{with probability } p; \\ u_i(t-1), & \text{with probability } 1-p. \end{cases}$$

where we let $u_0(1) = m + mp$, and $u_j(j) = m$ for all $j \geq 1$. The endpoints of the m edges emanating from vertex v_t are chosen independently from $\{v_0, v_1, \ldots, v_{t-1}\}$, and the probability that v_i is chosen as the endpoint of a fixed edge is equal to

$$\frac{u_i(t-1)}{\sum_{j=0}^{t-1} u_j(t-1)}. \tag{1}$$

After t steps following this growth procedure, there are $n_t = t + 1$ vertices and $e_t = mt$ edges. In our model, it is reasonable for us to consider the new vertex to be positive and its utility increased by 1 precisely, when one of its new edges is connected to an old vertex present in the networks, while the chosen vertex is seen to be passive and its utility is increased by 1 with probability p.

Remark 1. This model is closest in spirit to the model in Zheng et al. (2007). In fact, Zheng et al. (2007) do not give the rigorous description of their model. For example, they do not either give the concrete state of the initial graph $G(1)$, or give the rigorous mathematical analysis of their conclusions. In our model, we take a simple choice for the initial graph $G(1)$, and we will do our job in mathematical way rigorously.

1.2. Two useful lemmas

In our work, we need to use the following two outstanding lemmas, here we only state their contents without giving the proofs, for which the reader is referred to Chung and Lu (2006b). For $\mathbf{C} = (c_1, c_2, \ldots, c_n)$ a vector with positivity entries, the martingale X is said to be $\mathbf{C}$-Lipschitz if

$$|X_i - X_{i-1}| \leq c_i \tag{2}$$

for $i = 1, 2, \ldots, n$.

Lemma 1 (*Chung and Lu (2006b), Azuma's inequality*). *If a martingale X is $\mathbf{C}$-Lipschitz, then*

$$Pr(|X - E(X)| \geq \lambda) \leq 2 \exp^{-\frac{\lambda^2}{2\sum_{i=1}^{n} c_i^2}} \tag{3}$$

where $\mathbf{C} = (c_1, c_2, \ldots, c_n)$.

Lemma 2 (*Chung and Lu (2006b)*). *Suppose that a sequence $\{a_t\}$ satisfies the recurrence relation*

$$a_{t+1} = \left(1 - \frac{b_t}{t + t_1}\right) a_t + c_t \quad \text{for } t \geq t_0. \tag{4}$$

Furthermore, suppose $\lim_{t\to\infty} b_t = b > 0$ and $\lim_{t\to\infty} c_t = c$. Then $\lim_{t\to\infty} \frac{a_t}{t}$ exists and

$$\lim_{t\to\infty} \frac{a_t}{t} = \frac{c}{1+b}. \tag{5}$$