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Solutions for functional fully coupled forward–backward stochastic differential equations

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A B S T R A C T

In this paper, we study a functional fully coupled forward–backward stochastic differential equation (FBSDE). For this functional FBSDE, the classical Lipschitz and monotonicity conditions which guarantee the existence and uniqueness of the solution to FBSDE are no longer applicable. To overcome this difficulty, we propose a completely new type of Lipschitz and monotonicity condition in which an integral term with respect to the path of $X(t)_{0\leq t\leq T}$ is involved. Under this integral Lipschitz and monotonicity conditions, the existence and uniqueness of solutions for functional fully coupled FBSDEs is proved.

1. Introduction

Linear backward stochastic differential equations (in short BSDEs) was introduced by Bismut (1973). Pardoux and Peng (1990) established the existence and uniqueness theorem for nonlinear BSDEs under a standard Lipschitz condition. Since then, backward stochastic differential equations and forward–backward stochastic differential equations (FBSDEs) have been widely recognized that they provide useful tools in many fields, especially mathematical finance, the stochastic control theory (see Cvitanic and Ma, 1996; Ma and Yong, 1999; Yong, 2006, 2010 and the references therein) and stochastic delay systems (see Chang et al., 2008; Chen and Wu, 2012; Mohammed, 1984, 1996).

A state dependent fully coupled FBSDE is formulated as:

$$\begin{aligned} X(t) &= x + \int_0^t b(X(s), Y(s), Z(s))ds + \int_0^t \sigma(X(s), Y(s), Z(s))dW(s), \\ Y(t) &= g(X(T)) - \int_t^T h(X(s), Y(s), Z(s))ds - \int_t^T Z(s)dW(s), \quad t \in [0, T]. \end{aligned} \tag{1.1}$$

There have been three main methods to solve FBSDE (1.1), i.e., the Method of Contraction Mapping (see Antonelli, 1993; Pardoux and Tang, 1999), the Four Step Scheme (see Ma et al., 1994) and the Method of Continuation (see Hu and Peng, 1995; Peng and Wu, 1997; Yong, 1997). In Ma et al. (2015), Ma et al. studied the wellposedness of the FBSDEs in a general non-Markovian framework. They find a unified scheme which combines all existing methodology in the literature, and overcome some fundamental difficulties that have been long-standing problems for non-Markovian FBSDEs.

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In this paper, we study the following functional fully coupled FBSDE:

$$\begin{aligned} X(t) &= x + \int_0^t b(X_s, Y(s), Z(s))ds + \int_0^t \sigma(X_s, Y(s), Z(s))dW(s), \\ Y(t) &= g(X(T)) - \int_t^T h(X_s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s), \quad t \in [0, T], \end{aligned} \tag{1.2}$$

where $X_s := X(t)_{0\le t\le s}$.

As mentioned above, Hu and Peng (1995) initiated the continuation method in which the key issue is a certain monotonicity condition. But unfortunately, the Lipschitz and monotonicity conditions in Hu and Peng (1995) and Peng and Wu (1997) do not work for the functional equation (1.2). Here the main difficulty is that the coefficients of (1.2) depend on the path of the solution $X(t)_{0\le t\le T}$. To overcome this difficulty, we propose a completely new type of Lipschitz and monotonicity conditions. These new conditions involve an integral term with respect to the path of $X(t)_{0\le t\le T}$. Thus, we call them the integral Lipschitz and monotonicity conditions. The readers may see Assumptions 2.2 and 2.3 for more details. In particular, we present two examples to illustrate that our assumptions are not restrictive. Under the integral Lipschitz and monotonicity conditions, the continuation method for the functional equation (1.2) can go through and it leads to the existence and uniqueness of the solution to Eq. (1.2).

The paper is organized as follows. In Section 2, we formulate the problem and give the integral Lipschitz and monotonicity conditions. The existence and uniqueness of the solution for (1.2) is proved in the first part of Section 3.

2. Formulation of the problem

Let $\Omega = C([0, T]; \mathbb{R}^d)$ and P the Wiener measure on $(\Omega, \mathcal{B}(\Omega))$. We denote by $W = (W(t)_{t\in[0,T]})$ the canonical Wiener process, with $W(t, \omega) = \omega(t), t \in [0, T], \omega \in \Omega$. For any $t \in [0, T]$, we denote by $\mathcal{F}_t$ the P-completion of $\sigma(W(s), s \in [0, t])$.

For $n \in \mathbb{N}^+$, set

$$C_t^n = C([0, t]; \mathbb{R}^n) \quad \text{and} \quad C^n = \bigcup_{t\in[0,T]} C_t^n.$$

Consider the following functional fully coupled FBSDE:

$$X(t) = x + \int_0^t b(X_s, Y(s), Z(s))ds + \int_0^t \sigma(X_s, Y(s), Z(s))dW(s), \tag{2.1}$$

$$Y(t) = g(X(T)) - \int_t^T h(X_s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s), \quad t \in [0, T], \tag{2.2}$$

where the processes X, Y, Z take values in $\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^{m\times d}$, $X_s = X(r)_{0\le r\le s}$ and

$$\begin{aligned} &b\colon C^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \Omega \longrightarrow \mathbb{R}^n; \\ &\sigma\colon C^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \Omega \longrightarrow \mathbb{R}^{n\times d}; \\ &h\colon C^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \Omega \longrightarrow \mathbb{R}^m; \\ &g\colon \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^m. \end{aligned}$$

For $z \in \mathbb{R}^{m\times d}$, define $|z| = \{tr(zz^*)\}^{1/2}$, where "$*$" means transpose. For $z^1 \in \mathbb{R}^{m\times d}, z^2 \in \mathbb{R}^{m\times d}$,

$$((z^1, z^2)) \triangleq tr(z^1(z^2)^*).$$

We use the notations

$$\begin{aligned} &(x^1, u^1) = (x^1, y^1, z^1) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}; \\ &(x^2, u^2) = (x^2, y^2, z^2) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}; \\ &\langle\langle (x^1, u^1), (x^2, u^2)\rangle\rangle = \langle x^1, x^2\rangle + \langle y^1, y^2\rangle + ((z^1, z^2)). \end{aligned}$$

Given an $m \times n$ full-rank matrix G, for $(x_t, u) \in C^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$, define

$$f(x_t, u) = (G^* h(x_t, u), Gb(x_t, u), G\sigma(x_t, u)),$$

where $G\sigma = (G\sigma_1, \dots, G\sigma_d)$.

For any Hilbert space $(H, \|\cdot\|)$, we denote by $M^2(0, T; H)$ the set of all H-valued $\mathcal{F}_t$-adapted processes $\vartheta(\cdot)$ such that

$$E\int_0^T \|\vartheta(s)\|^2 ds < +\infty.$$

Definition 2.1. A triple $(X, Y, Z) : [0, T] \times \Omega \longrightarrow \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ is called an adapted solution of Eqs. (2.1) and (2.2), if $(X, Y, Z) \in M^2(0, T; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d})$, and it satisfies (2.1) and (2.2) P-a.s.