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The joint distribution of the maximum and minimum of an AR(1) process

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ABSTRACT

Consider a sequence of n observations from an autoregressive process of order 1 with maximum M_n and minimum m_n. We give their joint cumulative distribution function first in terms of n repeated integrals and then, for the case, where the marginal distribution of the observations is absolutely continuous, as a weighted sum of nth powers of eigenvalues of a certain Fredholm kernel. This enables good approximations for the joint distribution when n repeated integrals is not a practical solution.

1. Introduction

Given observations $X_1, X_2, \ldots, X_n$, let $M_n = \max(X_1, X_2, \ldots, X_n)$ and $m_n = \min(X_1, X_2, \ldots, X_n)$. The joint distribution of M_n and m_n is of interest in many applied areas: (i) annual maximum and annual minimum daily temperatures at a given city; (ii) annual maximum and annual minimum daily rainfall at a given city; (iii) annual maximum and annual minimum levels for a given river; (iv) maximum daily stock returns and minimum daily stock returns taken within a reference period; and so on.

There has not been much theory developed on the joint distribution of M_n and m_n. Bickel and Rosenblatt (1973) derived the asymptotic joint distribution of M_n and m_n for Gaussian processes, showing that they are asymptotically independent. Davis (1979) and Leadbetter et al. (1983) showed under certain general conditions that the asymptotic joint distribution of M_n and m_n is completely determined by its margins and that M_n and m_n are asymptotically independent. These results are not very useful in practice because n is always finite. There are many practical examples, where M_n and m_n are not significantly independent even for large n. We now give one example.

The data we consider are daily log returns of the S&P 500 stock value from the 3rd of January 2000 to the 28th of February 2014. The data were obtained from the database Datastream. A histogram of the data is shown in Fig. 1. We took maximums and minimums over non-overlapping blocks of width $n = 100$. A chisquare test of independence gave a p-value of 0.032.

This is just one example. One can find many more practical examples. Hence, the development of theory for the joint distribution of M_n and m_n for finite n is important. We are aware of no such developments. The aim of this note is to derive the *exact* joint distribution of M_n and m_n for finite n for autoregressive processes.

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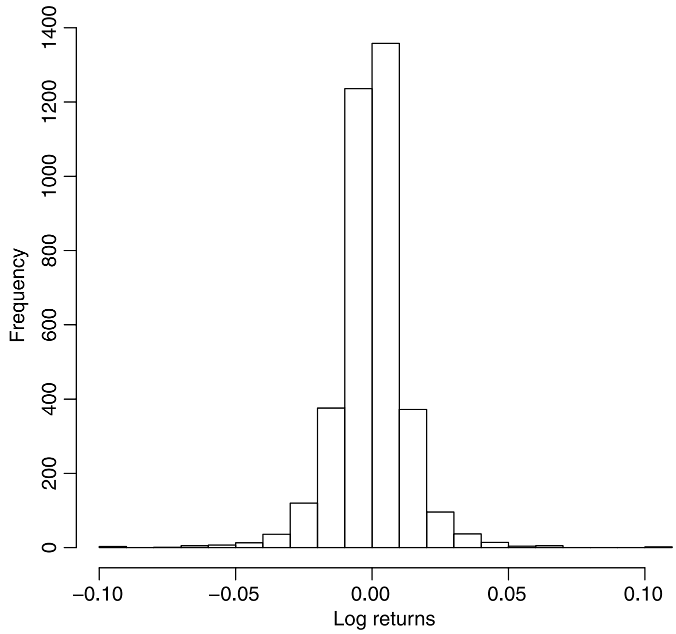

Fig. 1. Histogram of the log returns of the S&P 500 stock value.

However, the recent years have seen considerable developments on the asymptotics of the joint distribution of M_n and m_n following on from Davis (1979). We mention: the joint distribution of M_n and m_n for complete and incomplete samples of stationary sequences (Peng et al., 2010); the joint distribution of M_n and m_n for complete and incomplete samples from weakly dependent stationary sequences (Peng et al., 2011); the joint distribution of M_n and m_n for strongly dependent Gaussian vector sequences (Weng et al., 2012); the joint distribution of M_n and m_n for independent spherical processes (Hashorva, 2013); the joint distribution of M_n and m_n for dependent stationary Gaussian arrays (Hashorva and Weng, 2013); large deviation results on M_n and m_n for independent and identical samples (Giuliano and Macci, 2014); the joint distribution of M_n and m_n for scaled stationary Gaussian sequences (Hashorva et al., 2014); the joint distribution of M_n and m_n for complete and incomplete stationary sequences (Hashorva and Weng, 2014a,b); the joint distribution of M_n and m_n for Hüsler–Reiss bivariate Gaussian arrays (Liao and Peng, 2014). None of these papers give the exact joint distribution of M_n and m_n for finite n.

This note applies a powerful new method for giving the exact distribution of extremes of n correlated observations as weighted sums of nth powers of associated eigenvalues. The method was first illustrated by obtaining the distribution of the maximum of an autoregressive process of order 1 in Withers and Nadarajah (2011) then for the maximum of a moving average process of order 1 in Withers and Nadarajah (2014).

Let $\{e_i\}$ be independent and identically distributed random variables from some cumulative distribution function (cdf) F on $\mathbb{R}$. We consider the autoregressive process of order 1,

$$X_i = e_i + \rho X_{i-1},$$

where $\rho \neq 0$. So, $\mathrm{corr}\left(X_i, X_{i+j}\right) = \rho^j$. In Section 2, we give expressions for the joint cdf of M_n and m_n in terms of repeated integrals. This is obtained via the recurrence relationship

$$G_n(y) = \mathcal{K} G_{n-1}(y), \quad n \geq 1, \tag{1}$$

where

$$G_n(y) = P\left(x_1 < m_n, M_n \leq x_2, X_n \leq y\right), \tag{2}$$

$$\mathcal{K} h(y) = \mathrm{sign}(\rho) I\left\{x_1 < y\right\} \left(\mathcal{N} h\right)(y), \tag{3}$$

$$\left(\mathcal{N} h\right)(y) = \left(\mathcal{M} h\right)(y \wedge x_2) - \left(\mathcal{M} h\right)(x_1) = \left(\left(\Delta \mathcal{M}\right) h\right)(y) \text{ say},$$

$$y \wedge x_2 = \min\left(y, x_2\right),$$

and

$$\mathcal{M} h(y) = \mathbb{E}\left[h\left(\left(y - e_0\right)/\rho\right)\right],$$

where $I\{A\} = 1$ if A is true, $I\{A\} = 0$ if A is false and $x_1 < x_2$ are given. So, $\mathcal{K}$ is a linear integral operator depending on $\mathbf{x} = (x_1, x_2)$. Dependence on $\mathbf{x}$ is suppressed, where possible. For this to work at $n = 1$, we define $m_0 = \infty, M_0 = -\infty$ so that

$$G_0(y) = H(y),$$