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Sequential detection of gradual changes in the location of a general stochastic process

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ABSTRACT

We present monitoring procedures for a class of general stochastic processes which exhibit a possible gradually increasing or decreasing perturbation in their otherwise linear drift term. The suggested detectors are based on weighted increments of the process for which we derive asymptotic critical values and under the alternative consistency and asymptotic normality of the stopping times.

1. Introduction

Probably the most studied model in change point analysis is that of detecting a (single) abrupt jump in the otherwise constant mean of a stochastic process. In many applications, however, not a sudden but a gradual change, i.e. a slowly increasing or decreasing change, in the location of a process seems to be the far more realistic scenario. For instance (if no obvious shock has occurred) meteorological parameters, such as global temperature, elevation of lakes or average precipitation, are more likely to change slowly than abruptly. Still, testing procedures that are designed to detect gradual changes in the location of a process have gotten less attention in the literature.

Jarušková (1998) and Hušková (1998a,b) started to study models with linearly increasing changes in the mean, basing their inference on sums of weighted observations where a quasi maximum likelihood approach suggests to use the (assumed) type of change as a weighting. The idea is to discount the influence of the early observations – where a possible change has either not occurred (yet) or is still quite small – and enhance the influence of the late observations – where a possible change is at its current maximum – in order to obtain more reliable testing procedures. Similar approaches were further applied in more general settings e.g. by Hušková and Steinebach (2000), Steinebach (2000), Kirch and Steinebach (2006), Steinebach and Timmermann (2011) and Dette and Vogt (2014).

In this paper we consider a general stochastic process with a linear trend which exhibits a possible perturbation at some unknown time point, where we also allow for a possible change in the scale parameter. We aim to keep the setting as general as possible, focusing on the required invariance principles, yet, throughout this paper, Example 2.1 serves as a motivation. Note that is not necessary, yet reasonable, to base the weighting on the (typically unknown) type of change, which makes the suggested procedures highly applicable in practice.

The paper is organized as follows: In Section 2 we describe the testing problem for which we introduce suitable detectors and stopping times in Section 3. We present testing procedures for known and unknown "in-control"-parameters, i.e. the location- and scale parameters under the null hypothesis. Via the results under the null hypothesis of Section 4 one can obtain asymptotic critical values for the suggested procedures. Section 5 deals with the limiting behaviour under the alternative which includes the consistency of the tests (Section 5.1) and the asymptotic normality of the standardized delay times, i.e.

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the time lag between the change point and its detection (Section 5.2). All proofs are postponed to Section 6. The aim of the paper is to present results from the dissertation by Timmermann (2014). So, for the sake of clarity we only give some shortened proofs here and refer to the latter work for more details as well as for a small simulation study.

2. The testing problem

Assume, we sequentially observe a process

$$Z(t) = \begin{cases} bY(t) + at, & 0 \le t \le T^*, \\ bY(T^*) + b^* Y^*(t - T^*) + at + \Delta_{m,\gamma}(t - T^*), & T^* < t < \infty, \end{cases} \tag{2.1}$$

at integer time points where $\{Y(t) \mid t \ge 0\}$ and $\{Y^*(t) \mid t \ge 0\}$ are two stochastic processes for which the following invariance principles shall hold true: There are Wiener processes $\{W(t) \mid t \ge 0\}$ and $\{W^*(t) \mid t \ge 0\}$ and some $0 < \kappa < 1/2$ such that

$$\sup_{0<t<T^*} |Y(t) - W(t)| / t^\kappa = \mathcal{O}_P(1), \qquad \sup_{0<t<\infty} \left|Y^*(t) - W^*(t)\right| / (t + T^*)^\kappa = \mathcal{O}_P(1) \tag{2.2}$$

as $T^* \to \infty$. Further, $a \in \mathbb{R}$ and $b,\ b^* \in \mathbb{R}_+$ are (typically) unknown parameters. The unknown (deterministic) time point $T^* = T_m^*$ is called the change point and shall rely on the length m of a so-called "training period", i.e. an observation period during which we know that no change occurs. The assumption of such a training period is known as "non-contamination assumption" and formally states that $m < T^*$, for all $m \in \mathbb{N}$. The change $\Delta_{m,\gamma}(t - T^*)$ is a strictly in – or decreasing, deterministic function in $t \ge T^*$, possibly relying on further parameters (indicated by γ) and on the training period m. All asymptotic results presented in this paper rely on $m \to \infty$. Moreover, for the sake of simplicity, we assume $Y(0) = Y^*(0) = 0$ a.s. and $\Delta_{m,\gamma}(t - T^*) = 0$ for $t \le T^*$.

We are interested in testing the null hypothesis $H_0 : T^* = \infty$ "no change" against either one of the following alternatives

$H_1^+ : T^* < \infty,\ \Delta > 0$ "one-sided, positive change",

$H_1^- : T^* < \infty,\ \Delta < 0$ "one-sided, negative change",

$H_1 : T^* < \infty,\ \Delta \ne 0$ "two-sided change",

where by $\Delta > 0\ (< 0,\ \ne 0)$ we mean $\Delta_{m,\gamma}(t - T^*) > 0\ (< 0,\ \ne 0)$ for all $t > T^*$. Throughout this paper we will abbreviate the alternatives H_1^+ and H_1^- by $H_1^\pm$.

The following example will be considered in Section 5.2. Further examples can be found in e.g. following Horváth and Steinebach (2000) and Steinebach (2000).

Example 2.1. Assume we observe a process $X_i = \varepsilon_i + \mu + \delta_m \left((i - T^*)/m\right)_+^\gamma$, where $\mu,\ \gamma,\ \delta_m \in \mathbb{R}$, with $\gamma > 0$, $\delta_m \ne 0$, $x_+ := \max\{x, 0\}$ and $\{\varepsilon_i \mid i \in \mathbb{N}\}$ being i.i.d. random variables with $\mathrm{E}(\varepsilon_i) = 0$, $\mathrm{Var}(\varepsilon_i) = \sigma^2 > 0$ and $\mathrm{E}|\varepsilon_i|^{1/\kappa} < \infty$ for some $0 < \kappa < 1/2$. The parameter $\delta = \delta_m$ may particularly depend on m, where typically one is interested in so-called "local alternatives", i.e. $\delta_m \to 0$ as $m \to \infty$. On setting $a = \mu$, $b = b^* = \sigma$, $Y(t) = \sum_{i=1}^{[t]} \varepsilon_i$, $Y^*(t) = \sum_{i=[T^*]+1}^{[t]} \varepsilon_i$ and $\Delta_{m,\gamma}(t - T^*) = \sum_{i=1}^{[t]} \delta_m \left((i - T^*)/m\right)_+^\gamma$ we obtain a process in the form of (2.1). Further, by Komlós et al. (1975) we know that there is some Wiener process $\{W(t) \mid t \ge 0\}$ such that $|\sum_{i=1}^n \varepsilon_i - \sigma W(n)| \overset{\text{a.s.}}{=} \mathcal{O}\left(n^\kappa\right)$, hence on approximating $Y(t)$ by $W(t)$ and $Y^*(t)$ by $W^*(t) := W(t + T^*) - W(T^*)$ the assumptions on the invariance principles of (2.2) are fulfilled.

3. The stopping times

For our sequential testing procedure we are looking for stopping times τ_m such that under the null hypothesis it holds for some fixed $\alpha \in (0, 1)$ that $\lim_{m\to\infty} P(\tau_m < \infty) = \alpha$ and under the alternative it holds that $\lim_{m\to\infty} P(\tau_m < \infty) = 1$. The stopping times suggested below are based on detectors that, after each observation, determine whether a suitably chosen threshold function is exceeded (in which case the sequential procedure is stopped) or not (in which case the monitoring of the process is continued).

We standardize the observations by the in-control parameters, i.e. the location-parameter a and the scale-parameter b. If the in-control parameters are unknown, which is usually the case, they have to be estimated. Detectors and stopping times for known or unknown in-control parameters are introduced in Section 3.1 or Section 3.2, respectively.

3.1. Stopping times for known in-control parameters

We adopt the approach of Jarušková (1998) and Hušková (1998a) and construct our detectors as sums of weighted, standardized increments of the observed time series, putting the heaviest weight on the latest observation, where the "size" of a possible change is the largest. Denoting the increments by $Z_i = Z(i) - Z(i-1)$ we consider the following detectors:

$$T_k = \sum_{i=1}^{k} g(i/m)\,(Z_i - a)/(b\sqrt{m}), \quad k \ge m,$$