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Dilatively semistable stochastic processes☆

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ABSTRACT

Dilative semistability extends the notion of semi-selfsimilarity for infinitely divisible stochastic processes by introducing an additional scaling in the convolution exponent. It is shown that this scaling relation is a natural extension of dilative stability and some examples of dilatively semistable processes are given. We further characterize dilatively stable and dilatively semistable processes as limits for certain rescaled aggregations of independent processes.

1. Introduction

Let $\mathbb{T}$ be either $\mathbb{R}$, $[0,\infty)$ or $(0,\infty)$. Following Barczy et al. (in press) a stochastic process $(X_t)_{t\in\mathbb{T}}$ on $\mathbb{R}$ is called (α,δ)-*dilatively stable* for some parameters $\alpha,\delta\in\mathbb{R}$ if all its finite-dimensional marginal distributions are infinitely divisible and the scaling relation

$$\psi_{Tt_1,\dots,Tt_k}(\theta_1,\dots,\theta_k)=T^{\delta}\psi_{t_1,\dots,t_k}(T^{\alpha-\delta/2}\theta_1,\dots,T^{\alpha-\delta/2}\theta_1)$$

holds for all $T>0$, $k\in\mathbb{N}$, $\theta_1,\dots,\theta_k\in\mathbb{R}$, and $t_1,\dots,t_k\in\mathbb{T}$, where $\psi_{t_1,\dots,t_k}$ denotes the log-characteristic function of $(X_{t_1},\dots,X_{t_k})$, which is the unique continuous function with $\psi_{t_1,\dots,t_k}(0,\dots,0)=0$ fulfilling

$$\mathbb{E}\Big[\exp\Big(i\sum_{j=1}^{k}\theta_jX_{t_j}\Big)\Big]=\exp\left(\psi_{t_1,\dots,t_k}(\theta_1,\dots,\theta_k)\right).$$

This definition extends (Iglói, 2008) original formulation in the following way. Iglói additionally assumes $\mathbb{T}=[0,\infty)$, $X_0=0$, X_1 is non-Gaussian, and X_t has finite moments of arbitrary order for every $t\geq0$ in which case he was able to show that the parameters α,δ are uniquely determined and restricted to $\alpha>0$, $\delta\leq2\alpha$. We refuse to assume these additional conditions, since uniqueness of the parameters does not matter here. Roughly speaking, for $\delta\neq0$ dilative stability means

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that moving along the one-parameter semigroup $(\mu^s)_{s>0}$ generated by the finite-dimensional marginal distribution μ of $(X_{t_1},\dots,X_{t_k})$ coincides with the distribution of the space–time transformation $s^{\frac{1}{2}-\frac{\alpha}{\delta}}(X_{s^{1/\delta}t_1},\dots,X_{s^{1/\delta}t_k})$, whereas for $\delta=0$ dilative stability coincides with selfsimilarity. Note that Kaj (2005) introduced a weaker scaling relation called *aggregate-similarity*, which has been extended in Definition 1.4 of Barczy et al. (in press) such that dilative stability and aggregate similarity essentially define the same property if one additionally assumes infinite divisibility and weak right-continuity of the finite-dimensional marginal distributions; see Proposition 1.5 in Barczy et al. (in press) for details.

In Section 2 we will introduce a weaker scaling property called dilative semistability which naturally comes into play assuming weak continuity. This notion extends the class of infinitely divisible semi-selfsimilar processes introduced in Maejima and Sato (1999) also called discrete scale invariant processes in Borgnat et al. (2002). We give some examples of dilatively semistable process, in particular we point out how dilatively semistable generalized fractional Lévy motions can be constructed from dilatively stable counterparts of Barczy et al. (in press). Finally, in Section 3 we show that in a general limit procedure for certain aggregation models, dilatively stable and dilatively semistable processes can be characterized as limit processes.

2. Dilatively semistable processes

Let $X=(X_t)_{t\in\mathbb{T}}$ be a stochastic process on $\mathbb{R}$ whose finite-dimensional marginal distributions are infinitely divisible. Inspired by Urbanik's decomposability group in Urbanik (1972), for $\alpha,\delta\in\mathbb{R}$ we define the *dilative decomposability group* of X by

$$D_X(\alpha,\delta)=\left\{c>0:\begin{array}{c}\psi_{ct_1,\dots,ct_k}(\theta_1,\dots,\theta_k)=c^{\delta}\psi_{t_1,\dots,t_k}(c^{\alpha-\delta/2}\theta_1,\dots,c^{\alpha-\delta/2}\theta_k)\\ \text{for all } k\in\mathbb{N},\theta_1,\dots,\theta_k\in\mathbb{R},\text{ and } t_1,\dots,t_k\in\mathbb{T}\end{array}\right\},$$

where $\psi_{t_1,\dots,t_k}$ again denotes the log-characteristic function of $(X_{t_1},\dots,X_{t_k})$ and the notion "group" is justified as follows.

Proposition 2.1. *If the finite-dimensional distributions of X are weakly continuous, then $D_X(\alpha,\delta)$ is a closed subgroup of $\mathbb{G}=((0,\infty),\cdot)$.*

Proof. If $b,c\in D_X(\alpha,\delta)$, we have

$$\begin{aligned}\psi_{bct_1,\dots,bct_k}(\theta_1,\dots,\theta_k)&=b^{\delta}\psi_{ct_1,\dots,ct_k}(b^{\alpha-\delta/2}\theta_1,\dots,b^{\alpha-\delta/2}\theta_k)\\&=(bc)^{\delta}\psi_{t_1,\dots,t_k}((bc)^{\alpha-\delta/2}\theta_1,\dots,(bc)^{\alpha-\delta/2}\theta_k)\end{aligned}$$

showing that $bc\in D_X(\alpha,\delta)$. Hence $D_X(\alpha,\delta)$ is a subgroup of $\mathbb{G}$. If $c_n\in D_X(\alpha,\delta),\ n\in\mathbb{N}$, is a sequence with $c_n\to c>0$ then our assumption on weak continuity implies

$$\begin{aligned}\psi_{ct_1,\dots,ct_k}(\theta_1,\dots,\theta_k)&=\lim_{n\to\infty}\psi_{c_nt_1,\dots,c_nt_k}(\theta_1,\dots,\theta_k)\\&=\lim_{n\to\infty}c_n^{\delta}\psi_{t_1,\dots,t_k}(c_n^{\alpha-\delta/2}\theta_1,\dots,c_n^{\alpha-\delta/2}\theta_k)\\&=c^{\delta}\psi_{t_1,\dots,t_k}(c^{\alpha-\delta/2}\theta_1,\dots,c^{\alpha-\delta/2}\theta_k)\end{aligned}$$

showing that $c\in D_X(\alpha,\delta)$. Hence $D_X(\alpha,\delta)$ is a closed subgroup of $\mathbb{G}$.

Since the only non-trivial closed subgroups are $\mathbb{G}$ itself (leading to dilative stability) and $c^{\mathbb{Z}}=\{c^m:m\in\mathbb{Z}\}$ for some $c>1$, the following property naturally appears.

Definition 2.2. A stochastic process $X=(X_t)_{t\in\mathbb{T}}$ is said to be (c,α,δ)-*dilatively semistable* for parameters $c>1$ and $\alpha,\delta\in\mathbb{R}$ if its finite-dimensional marginal distributions are infinitely divisible and $c^{\mathbb{Z}}\subseteq D_X(\alpha,\delta)$.

Examples 2.3. (a) By Definition 2.2, any (α,δ)-dilatively stable process is also (c,α,δ)-dilatively semistable for every $c>1$.

Conversely, let $X=(X_t)_{t\in\mathbb{T}}$ be a weakly continuous (b,α,δ) and (c,α,δ)-dilatively semistable process, where $b,c>1$ are exponentially incommensurable in the sense that $b^n\neq c^m$ for all $n,m\in\mathbb{Z}$. Then Proposition 2.1 yields $(0,\infty)=\overline{\{b^nc^m:n,m\in\mathbb{Z}\}}\subseteq D_X(\alpha,\delta)$ showing that X is (α,δ)-dilatively stable.

(b) Let $X=(X_t)_{t\ge0}$ be a semi-selfsimilar process with Hurst index $H>0$, i.e.

$$(X_{ct})_{t\ge0}\overset{\text{fd}}{=}(c^HX_t)_{t\ge0}\quad\text{for some } c>1,$$

where "$\overset{\text{fd}}{=}$" denotes equality in distribution of all finite-dimensional marginal distributions. Then obviously X fulfills the scaling property of a $(c,H,0)$-dilatively semistable process for which (due to $\delta=0$) infinite divisibility is not needed. Hence dilative semistability extends semi-selfsimilarity for infinitely divisible processes.

(c) Let $X=(X_t)_{t\ge0}$ be a (c,γ)-semistable Lévy process, i.e. a semi-selfsimilar Lévy process with Hurst index $H=1/\gamma$ for some $c>1$ and $\gamma\in(0,2)$. Then by semi-selfsimilarity we have

$$\psi_{ct_1,\dots,ct_k}(\theta_1,\dots,\theta_k)=\psi_{t_1,\dots,t_k}(c^{1/\gamma}\theta_1,\dots,c^{1/\gamma}\theta_k)$$