Statistics and Probability Letters 99 (2015) 223–229

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Limit theorems for discrete Hawkes processes

Youngsoo Seol

Department of Mathematics, Texas State University-SM, San Marcos, TX 78666, USA

ARTICLE INFO

Article history:
Received 23 December 2014
Received in revised form 14 January 2015
Accepted 18 January 2015
Available online 29 January 2015

MSC:
primary 60G55
secondary 60F05
60F17

Keywords:
Discrete Hawkes processes
Self-exciting processes
Central limit theorems
Law of large numbers
Invariance principles

ABSTRACT

We consider discrete time Hawkes process which is a class of g-functions. The limit theorems for continuous time Hawkes processes are well known and studied by many authors. In this paper, we study the limit theorems for discrete time Hawkes processes. In particular, we obtain the law of large number, the central limit theorem and the invariance principle.

1. Introduction

Hawkes process is a self-exciting simple point process introduced by Hawkes (1971). There has been tremendous progress in the study of continuous time Hawkes processes for both linear and nonlinear, however not much is known about discrete time Hawkes processes. Hawkes (1971) first studied for the linear case, which has an immigration-birth presentation. The stability, law of large numbers, central limit theorem, large deviations, Bartlett spectrum etc. have all been studied and understood very well. Almost all of the applications of Hawkes process in the literatures consider exclusively the linear case. Because of the lack of immigration-birth representation and computational tractability, nonlinear Hawkes process is much less studied. However, some efforts have already been made in this direction. Nonlinear Hawkes process was first introduced by Brémaud and Massoulié (1996). Recently, Zhu (2014a, 2013a,b,c) investigated several results for both linear and nonlinear model. The central limit theorem for nonlinear Hawkes processes was studied in Zhu (2013a) and the large deviation principles for nonlinear Hawkes processes have been obtained in Zhu (2014c). Hawkes process has been applied to many other fields including neuroscience, seismology, DNA modeling, finance. It has both self-exciting and clustering properties, which is very appealing to some financial applications. In particular, self-exciting and clustering properties of Hawkes process make it a viable candidate in modeling the correlated defaults and evaluating the credit derivatives in finance, for example, see Errais et al. (2010) and Dassios and Zhao (2011). Zhu (2013c) also have been studied for applications to financial mathematics. Some variations and extensions of Hawkes process have been studied in Dassios and Zhao (2011), Zhu (2014b) and Mehrdad and Zhu (2015).

Here is a general description of continuous time Hawkes processes.

E-mail address: prosul76@gmail.com.

http://dx.doi.org/10.1016/j.spl.2015.01.023
0167-7152/

Let N be a simple point process on $\mathbb{R}$ and let $\mathcal{F}_t^{-\infty} := \sigma(N(C), C \in \mathcal{B}(\mathbb{R}), C \in (-\infty, t])$ be an increasing family of σ-algebras. Any nonnegative $\mathcal{F}_t^{-\infty}$-progressively measurable process λ_t with

$$\mathbb{E}\left[N(a,b]|\mathcal{F}_a^{-\infty}\right] = \mathbb{E}\left[\int_a^b \lambda_s ds|\mathcal{F}_a^{-\infty}\right]$$

a. s. for all interval $(a, b]$ is called an $\mathcal{F}_t^{-\infty}$-intensity of N. We use the notation $N_t := N(0, t]$ to denote the number of points in the interval $(0, t]$.

A general Hawkes process is a simple point process N admitting an $\mathcal{F}_t^{-\infty}$-intensity

$$\lambda_t := \lambda\left(\int_{-\infty}^t h(t-s)N(ds)\right),$$

where $\lambda(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is locally integrable and left continuous, $h(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ and we always assume that $\|h\|_{L^1} = \int_0^\infty h(t)dt < \infty$. Here $\int_{-\infty}^t h(t-s)N(ds)$ stands for $\int_{(-\infty,t)} h(t-s)N(ds)$. We always assume that $N(-\infty, 0] = 0$, i.e. the Hawkes process has empty history. In the literatures, $h(\cdot)$ and $\lambda(\cdot)$ are usually referred to as exciting function and rate function respectively. The Hawkes process is linear if $\lambda(\cdot)$ is linear and it is nonlinear otherwise. In general, the model described above is non-Markovian since the future evolution of a self-exciting simple point process is controlled by the timing of past events but it is Markovian for a special case.

In this paper, we consider the discrete time Hawkes process S_n which is the special case of the g-functions (e.g. see Berbee, 1987; Bramson and Kalikow, 1993; Doeblin and Fortet, 1937; Keane, 1972) and study the limit theorems for the process S_n.

The structure of this paper is organized as follows. Some auxiliary results to prove the main results are stated in Section 2 and the main results in Section 3. The proofs for the main theorems are contained in Section 4.

2. Preliminaries

In this section we are setting for main problems and introduce the classical results. We start with some reviews for the results of continuous time Hawkes processes.

2.1. Limit theorems for continuous time Hawkes processes

The limit theorems for both linear and nonlinear continuous time Hawkes processes are well known and studied by many authors.

Linear continuous time model: Since $\lambda(\cdot)$ is linear, say $\lambda(z) = \nu + z$ for some $\nu > 0$, and $\|h\|_{L^1} < 1$, we can use a very nice immigration-birth representation and the limit theorems are well understood and more explicitly represented. Daley and Vere-Jones (2003) proved the law of large numbers for linear Hawkes process. The functional central limit theorem for linear multivariate Hawkes process under certain assumptions have been obtained by Bacry et al. (2012). Bordenave and Torrisi (2007) proved that if $0 < \mu < 1$ and $\int_0^\infty th(t)dt < \infty$, then $(\frac{N_t}{t} \in \cdot)$ satisfies the large deviation principle with the good rate function $I(x)$. Moderate deviation principle for linear continuous time Hawkes processes is obtained by Zhu (2013b) and the limit theorems for linear marked Hawkes processes are obtained in Karabash and Zhu (2015).

Nonlinear continuous time model: Since $\lambda(\cdot)$ is nonlinear, the usual immigration-birth representation no longer works and so nonlinear model is much harder to study. Brémaud and Massoulié (1996) proved that there exists a unique stationary version of nonlinear Hawkes processes under certain conditions and the convergence to equilibrium of a non-stationary version. Central limit theorem is obtained in Zhu (2013a) and Zhu (2014c) proved large deviation for a special case for nonlinear case when $h(\cdot)$ is exponential or sums of exponentials. Zhu (2014a) proved a process-level, i.e. level-3 large deviation principle for nonlinear Hawkes processes for general $h(\cdot)$ and hence by contradiction principle, the level-1 large deviation principle for $(\frac{N_t}{t} \in \cdot)$.

2.2. The discrete model

We will now describe in more detail a specific model of discrete case that we consider in this paper. We start with the assumptions we will use through the paper.

Assumption 2.1. Let $\hat{\mathbb{N}} = \mathbb{N} \cup \{0\}$

(A1) For $i \in \hat{\mathbb{N}}$, $\alpha_i > 0$ is a given sequence of positive numbers.
(A2) $\sum_{i=0}^{\infty} \alpha_i < 1$.

Assumption 2.2. (B1) $\sqrt{n}\sum_{i=n}^{\infty} \alpha_i \to 0$ as $n \to \infty$.
(B2) $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} i\alpha_i \to 0$ as $n \to \infty$.