Statistics and Probability Letters 98 (2015) 50–53

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

First hitting time of the integer lattice by symmetric stable processes

Yasuki Isozaki *

Department of Mathematical and Physical Science, Kyoto Institute of Technology, Matsugasaki, Sakyo-ku, Kyoto 606-8585, Japan

ARTICLE INFO

Article history:
Received 12 August 2014
Received in revised form 10 December 2014
Accepted 11 December 2014
Available online 18 December 2014

Dedicated to the memory of Professor Marc Yor

MSC:
60G52
60G51
60J45

Keywords:
Symmetric stable processes
Hitting time
Poisson summation formula

ABSTRACT

For one-dimensional Brownian motion, the first hitting time of a point has infinite mean while the exit time from an interval has finite exponential moments. In this note we establish its counterparts for symmetric stable processes. The Laplace transform of the first hitting time of the integer lattice is obtained.

1. Introduction and the main theorem

Let $B(t)$ be a one-dimensional Brownian motion starting from x. Its probability and expectation are denoted by P_x^{BM} and E_x^{BM}, respectively. For $a \in \mathbb{R}$, T_a denotes the first hitting time of a : $\inf\{t > 0 | B(t) = a\}$.

The density of the distribution of T_0 under P_x^{BM} is known and it holds $P_x^{BM}[T_0 > t] \sim \sqrt{2/\pi}|x|t^{-1/2}$ as $t \to \infty$ where $\sim$ means that the ratio between both sides converges to 1. Consequently, $E_x^{BM}[T_0] = \infty$.

Let $1 < \alpha < 2$, $C > 0$, and let $X(t)$ be a symmetric α-stable Lévy process starting from $x \in \mathbb{R}$ with the characteristic exponent $\Psi(\xi) = C|\xi|^\alpha$. Its probability and expectation are denoted by P_x and E_x, respectively. For example, $E_x[e^{i\xi X(t)}] = e^{i\xi x - \Psi(\xi)}$. By some abuse of notation, we denote by T_a the first hitting time of a by $X(t)$. It holds $E_x[T_0] = \infty$ if $x \neq 0$ and more precisely, $P_x[T_0 > t] \sim C_1(\alpha, C)|x|^{\alpha-1}t^{-1+1/\alpha}$ as $t \to \infty$. See Port (1967, Theorem 2) for the constant $C_1(\alpha, C)$. This is clearly a stable counterpart of the estimate of $P_x^{BM}[T_0 > t]$.

Let us next consider the exit time from an interval for $B(t)$. If $L > 0$, $0 < x < L$ and $B(0) = x$ then $T_0 \wedge T_L$ is the exit time from $[0, L]$. Its density is known in the form of a series expansion (Borodin and Salminen, 1996, p. 172 and p. 451) and the Laplace transform is obtained in a closed form: $E_x^{BM}[e^{-q(T_0 \wedge T_L)}] = \frac{\cosh(\sqrt{2q}(x-L/2))}{\cosh(\sqrt{2q}L/2)}$ for $q > 0$. By the analytic continuation $E_x^{BM}[e^{q(T_0 \wedge T_L)}] = \frac{\cos(\sqrt{2q}(x-L/2))}{\cos(\sqrt{2q}L/2)}$ is finite for $q \in [0, \frac{\pi^2}{2L^2})$. This feature is not shared by $E_x[e^{q(T_0 \wedge T_L)}]$, in fact, $E_x[T_0 \wedge T_L] = \infty$ can

* Tel.: +81 75 724 7359; fax: +81 75 724 7210.
E-mail address: isozaki@kit.ac.jp.

http://dx.doi.org/10.1016/j.spl.2014.12.013
0167-7152/

be derived from Proposition 5.4 of Yano et al. (2009). However, the first hitting time $T_{L\mathbb{Z}}$ of the set $L\mathbb{Z} = \{Lm | m \in \mathbb{Z}\}$ makes some sense. If $0 < x < L$ the law of $T_{L\mathbb{Z}}$ under P_x^{BM} is the same as that of $T_0 \wedge T_L$. On the other hand, under P_x, $T_{L\mathbb{Z}}$ has some finite exponential moments as is shown in the main theorem of this note.

Theorem 1. *Let $1 < \alpha < 2,\ L > 0,\ C > 0$, and $T_{L\mathbb{Z}} = \inf\{t > 0 | X(t) \in L\mathbb{Z}\}$ where $X(t)$ is a symmetric α-stable Lévy process starting from $x \in \mathbb{R}$ with the characteristic exponent $\Psi(\xi) = C|\xi|^\alpha$.*

(a) *For any $q > 0$ and $x \in \mathbb{R}$,*

$$E_x[e^{-qT_{L\mathbb{Z}}}] = \frac{1 + \sum_{n\in\mathbb{Z},\ n\neq 0} \frac{q}{q + C|2\pi n/L|^\alpha} e^{2\pi i n x/L}}{1 + \sum_{n\in\mathbb{Z},\ n\neq 0} \frac{q}{q + C|2\pi n/L|^\alpha}}. \tag{1}$$

(b) *It holds*

$$E_x[T_{L\mathbb{Z}}] = \sum_{n\in\mathbb{Z},\ n\neq 0} \frac{1}{C\,|2\pi n/L|^\alpha} \left(1 - e^{2\pi i n x/L}\right). \tag{2}$$

(c) *There exists $\rho > 0$ such that both sides of (1) are finite and the equality holds for any $q \in (-\rho, 0]$ and $x \in \mathbb{R}$. The radius ρ of convergence depends on C, α, L but not on x.*

Remark 1. The Fourier series expansion of the function $E_x^{BM}[e^{-q(T_0\wedge T_L)}] = \frac{\cosh(\sqrt{2q}(x-L/2))}{\cosh(\sqrt{2q}L/2)}$ of x on $[0, L]$ is $\frac{\sqrt{2}\tanh(\sqrt{2q}L/2)}{L\sqrt{q}} \sum_{n\in\mathbb{Z}} \frac{q}{q+\frac{1}{2}|2\pi n/L|^2} e^{2\pi i n x/L}$ if $q > 0$. Since the latter is a periodic function, it is equal to $E_x^{BM}[e^{-qT_{L\mathbb{Z}}}]$ on $\mathbb{R}$. The coincidence with Eq. (1) with the parameter $C = 1/2$ and $\alpha = 2$ can be checked using the identity $\frac{L\sqrt{q}}{\sqrt{2}} \coth(\sqrt{2q}L/2) = \sum_{n\in\mathbb{Z}} \frac{q}{q+\frac{1}{2}|2\pi n/L|^2}$.

Remark 2. Using the identities $\sum_{1\le n:\text{odd}} n\sin(nx)/(n^2 + a^2) = \pi(\sinh a(\pi - x) + \sinh ax)/(4\sinh a\pi)$, valid for $a \neq 0$ and $0 < x < \pi$, and $\frac{\cosh(\sqrt{2q}(x-L/2))}{\cosh(\sqrt{2q}L/2)} = \frac{\sinh(\sqrt{2q}(L-x))+\sinh(\sqrt{2q}x)}{\sinh(\sqrt{2q}L)}$, we can check the following partial fraction expansion in a complex variable q:

$$E_x^{BM}[e^{-qT_{L\mathbb{Z}}}] = (2\pi/L^2) \sum_{1\le n:\text{odd}} n\sin(n\pi|x|/L)/(q + n^2\pi^2/(2L^2)).$$

By the inverse Laplace transform we obtain the probability density of $T_{L\mathbb{Z}}$:

$$P_x^{BM}[T_{L\mathbb{Z}} \in dt]/dt = (2\pi/L^2) \sum_{1\le n:\text{odd}} n\sin(n\pi|x|/L)\exp(-n^2\pi^2 t/(2L^2)).$$

Note that this density can be derived via the spectral representation in Borodin and Salminen (1996, p. 105). Unfortunately we have not obtained analogous partial fraction expansion of $E_x[e^{-qT_{L\mathbb{Z}}}]$ nor a formula for the density (see Remark 3 in Section 2 for its existence) of $T_{L\mathbb{Z}}$ for $X(t)$.

2. Proof of the theorem

In Section 2 we first review some facts from the potential theory (see e.g. Bertoin, 1996, Chapter II). We next prove or quote preliminary lemmas before the proof of Theorem 1.

Since $1 < \alpha < 2$ a single point is regular for itself for $X(t)$. Let $q > 0$. The q-potential $u^q(x)$ is continuous and satisfies

$$u^q(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} \frac{1}{q + C|\xi|^\alpha} d\xi \tag{3}$$

and $E_x[e^{-qT_y}] = u^q(y-x)/u^q(0)$ for all x and y. The capacitary measure $\mu_{L\mathbb{Z}}^q$ is defined by $\mu_{L\mathbb{Z}}^q(A) = q\int_{-\infty}^{\infty} E_x[e^{-qT_{L\mathbb{Z}}}; X(T_{L\mathbb{Z}}) \in A]dx$ for any Borel set $A \subset \mathbb{R}$. Since the set $L\mathbb{Z}$ is translation invariant, $\mu_{L\mathbb{Z}}^q$ assigns the same mass $\mu_{L\mathbb{Z}}^q(\{0\})$ for each point of $L\mathbb{Z}$. By Theorem II.7 in Bertoin (1996) we have, for a.e. x,

$$E_x[e^{-qT_{L\mathbb{Z}}}] = \mu_{L\mathbb{Z}}^q(\{0\}) \sum_{m\in\mathbb{Z}} u^q(Lm - x). \tag{4}$$

Lemma 1. *For any $q > 0$ the following holds.*

(a) *As $x \to \pm\infty$, $u^q(x) = O(|x|^{-\alpha})$.*

(b) *The function $x \mapsto E_x[e^{-qT_{L\mathbb{Z}}}]$ is continuous.*

(c) *Eq. (4) holds for all x.*

Proof. (a) The integrand in (3) is of class C^1 and integration by parts applies to (3). We have

$$u^q(x) = \frac{1}{2\pi i x} \int_{-\infty}^{\infty} e^{-ix\xi} \frac{C\alpha|\xi|^{\alpha-1}\operatorname{sgn}(\xi)}{(q + C|\xi|^\alpha)^2} d\xi.$$