Statistics and Probability Letters 98 (2015) 54–58

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

# Markov limit of line of decent types in a multitype supercritical branching process

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ARTICLE INFO

*Article history:*
Received 31 July 2014
Received in revised form 22 October 2014
Accepted 13 November 2014
Available online 12 December 2014

*MSC:*
primary 60J80
secondary 60G50

*Keywords:*
Branching processes
Multitype
Supercritical
Markov

ABSTRACT

In a multitype ($d$ types) supercritical positively regular Galton–Watson branching process, let $\{X_n, X_{n-1}, \ldots, X_0\}$ denote the types of a randomly chosen (i.e., uniform distribution) individual from the $n$th generation and this individual's $n$ ancestors. It is shown here that this sequence converges in distribution to a Markov chain $\{Y_0, Y_1, \ldots\}$ with transition probability matrix $(p_{ij})_{1\le i,j\le d}$ and having the stationary distribution. We also consider the critical case conditioned on non-extinction.



## 1. Introduction

Let $\mathbf{Z}_n = (Z_{n,1}, Z_{n,2}, \ldots, Z_{n,d})$ be the population vector in the $n$th generation, $n = 0, 1, 2, \ldots$, where $Z_{n,i}$ is the number of individuals of type $i$ in the $n$th generation. We assume that each individual of type $i$, $i = 1, 2, \ldots, d$, lives a unit of time and, upon death, produces children of all types according to the offspring distribution $\left\{p^{(i)}(\mathbf{j}) \equiv p^{(i)}(j_1, j_2, \ldots, j_d)\right\}_{\mathbf{j}\in\mathbb{N}^d}$ and independently of other individuals, where $p^{(i)}(j_1, j_2, \ldots, j_d)$ is the probability that a type $i$ parent produces $j_1$ children of type $1, j_2$ children of type $2, \ldots, j_d$ children of type $d$.

Let $m_{ij} = E(Z_{1,j}|\mathbf{Z}_0 = \mathbf{e}_i)$ be the expected number of type $j$ offspring of a single type $i$ individual in one generation for any $i, j = 1, 2, \ldots, d$. Then,

$$\mathbf{M} \equiv \{m_{ij} : i, j = 1, 2, \ldots, d\} \tag{1.1}$$

is called the offspring mean matrix.

In a discrete-time multi-type positively regular Galton–Watson branching process, by the Perron–Frobenius theorem, the matrix $\mathbf{M}$ has a maximal eigenvalue $\rho$ and has associated strictly positive normalized right and left eigenvectors $\mathbf{u} = (u_1, u_2, \ldots, u_d)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_d)$ such that

$$\mathbf{u}\cdot\mathbf{v} = 1 \quad \text{and} \quad \mathbf{u}\cdot\mathbf{1} = 1. \tag{1.2}$$

The maximal eigenvalue $\rho$ of the offspring mean matrix $\mathbf{M}$ plays a crucial role. The process is called a supercritical, critical or subcritical branching process according as $1 < \rho < \infty$, $\rho = 1$ or $\rho < 1$, respectively (see Athreya and Ney, 2004 for the details).

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http://dx.doi.org/10.1016/j.spl.2014.11.013

Now, we consider Galton–Watson branching process with a finite offspring mean matrix **M** whose maximal eigenvalue $1 < \rho < \infty$ and with no extinction. Then choose an individual at random, i.e., uniform distribution, from the $n$th generation and denote its type $X_n$. Let $X_{n-1}$ be the type of its parent, $X_{n-2}$ the type of its grandparent, etc., down to $X_0$ being the type of the ancestor in the generation 0. We show that, for any integer $k$, $(X_n, X_{n-1}, \dots, X_{n-k})$ converges in distribution to $(Y_0, Y_1, \dots, Y_k)$ where $\{Y_n\}_{n\geq 0}$ is a Markov chain with a unique stationary distribution.

The work of Jagers and Nerman (1996) considers a similar process in a more general setting. However, their principal assumption is that the process has been evolving for an infinite amount of time and is already in a stable state. In this paper, we consider the case when the population has evolving up to $n$ generations and prove a limit result about the types of the ancestors of a random chosen individual as $n \to \infty$. Thus, the work reported here is related to but different from that in Jagers and Nerman (1996).

## 2. Main results

The first result is for the supercritical case. Without lose of generality, we assume that each individual in this supercritical process produces at least one offspring with probability 1 upon death, that is, $P(\mathbf{Z}_1 = \mathbf{0} | \mathbf{Z}_0 = \mathbf{e}_i) = 0$ for all $i = 1, 2, \dots, d$. Thus, the probability of extinction is 0.

**Theorem 2.1.** *Let $1 < \rho < \infty$, $|\mathbf{Z}_0| = 1$, $E\big(\|Z_1\| \log \|Z_1\| \big| \mathbf{Z}_0 = \mathbf{e}_i\big) < \infty$ for any $i = 1, 2, \dots, d$. Then, for any integer $k \geq 0$, there exists a random vector $(Y_0, Y_1, \dots, Y_k)$ such that*

$$\big(X_n, X_{n-1}, \dots, X_{n-k}\big) \xrightarrow{d} \big(Y_0, Y_1, \dots, Y_k\big) \quad \text{as } n \to \infty,$$

*and, for any $i_0, i_1, \dots, i_k \in \{1, 2, \dots, d\}$,*

$$P\big(Y_0 = i_0, Y_1 = i_1, \dots, Y_k = i_k\big) = \frac{v_{i_k} m_{i_k i_{k-1}} m_{i_{k-1} i_{k-2}} \cdots m_{i_1 i_0}}{(\mathbf{1} \cdot \mathbf{v}) \rho^k}.$$

*Moreover, $\{Y_n\}_{n\geq 0}$ is a Markov chain with the state space $\{1, 2, \dots, d\}$,*

(a) *initial distribution $\boldsymbol{\lambda}_0 \equiv \big(\lambda_0(1), \lambda_0(2), \dots, \lambda_0(d)\big)$ where*

$$\lambda_0(i) = \frac{v_i}{\mathbf{1} \cdot \mathbf{v}} \quad \text{for any } i = 1, 2, \dots, d,$$

(b) *transition probability $\mathbf{P} \equiv \big(p_{ij} : i, j = 1, 2, \dots, d\big)$, where*

$$p_{ij} = \frac{v_j m_{ji}}{v_i \rho} \quad \text{for any } n = 0, 1, 2, \dots,$$

(c) *and a unique stationary distribution $\boldsymbol{\pi} \equiv \big(\pi_1, \pi_2 \dots, \pi_d\big)$ where*

$$\pi_i = \frac{u_i v_i}{\mathbf{u} \cdot \mathbf{v}} \quad \text{for any } i = 1, 2, \dots, d.$$

A similar result also holds for the critical case conditioned on non-extinction:

**Theorem 2.2.** *Let $\rho = 1$, $|\mathbf{Z}_0| = 1$ and $E\|Z_1\|^2 < \infty$. Then, for any integer $k \geq 0$, there exists a random vector $(Y_0, Y_1, \dots, Y_k)$ such that*

$$\big(X_n, X_{n-1}, \dots, X_{n-k}\big) \Big| |\mathbf{Z}_n| > 0 \xrightarrow{d} \big(Y_0, Y_1, \dots, Y_k\big) \quad \text{as } n \to \infty,$$

*and, for any $i_0, i_1, \dots, i_k \in \{1, 2, \dots, d\}$,*

$$P\big(Y_0 = i_0, Y_1 = i_1, \dots, Y_k = i_k\big) = \frac{v_{i_k} m_{i_k i_{k-1}} m_{i_{k-1} i_{k-2}} \cdots m_{i_1 i_0}}{(\mathbf{1} \cdot \mathbf{v})}.$$

*Moreover, $\{Y_n\}_{n\geq 0}$ is a Markov chain with the state space $\{1, 2, \dots, d\}$,*

(a) *initial distribution $\boldsymbol{\lambda}_0 \equiv \big(\lambda_0(1), \lambda_0(2), \dots, \lambda_0(d)\big)$ where*

$$\lambda_0(i) = \frac{v_i}{\mathbf{1} \cdot \mathbf{v}} \quad \text{for any } i = 1, 2, \dots, d,$$

(b) *transition probability $\mathbf{P} \equiv \big(p_{ij} : i, j = 1, 2, \dots, d\big)$, where*

$$p_{ij} = \frac{v_j m_{ji}}{v_i} \quad \text{for any } n = 0, 1, 2, \dots,$$

(c) *and a unique stationary distribution $\boldsymbol{\pi} \equiv \big(\pi_1, \pi_2 \dots, \pi_d\big)$ where*

$$\pi_i = \frac{u_i v_i}{\mathbf{u} \cdot \mathbf{v}} \quad \text{for any } i = 1, 2, \dots, d.$$