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A residual-based test for variance components in linear mixed models

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ABSTRACT

In the paper, a residual-based test is developed for variance components in LMM. It is distribution-free, tractable, consistent and suitable for clustered designs and crossed designs. Numerical analysis is also conducted.

1. Introduction

Consider the following linear mixed model (LMM)

$$Y = X\beta + Zb + \epsilon \tag{1}$$

where Y is the observed $n \times 1$ response, X is the observed $n \times p$ covariates associated with the unknown $p \times 1$ regression coefficient β, Z is the observed $n \times q$ covariates associated with the unobserved $q \times 1$ random effect b which has mean zero and the unknown covariance matrix D, and ϵ is the $n \times 1$ random error vector with independent elements with mean zero and the variance σ^2.

Since the mean of the random effects is zero, testing whether the existence of random effects is equivalent to testing the following hypothesis:

$$H_0 : D = 0 \quad \text{versus} \quad H_A : D \geq 0 \quad \text{and} \quad D \neq 0. \tag{2}$$

(2) is a nonstandard testing problem since the true value is on the boundary of the parameter space. In fact, there are many literatures about testing the nullity of variance components in mixed models. In LMM for longitudinal data with the normality assumption about random effects and random errors, Stram and Lee (1994) considered likelihood ratio tests (LRTs) where the asymptotic distributions are weighted χ^2 distribution with the weights difficult to determine and the critical values must resort to Monte Carlo (MC) method when the dimension of the tested random effect is larger than one; Giampaoli and Singer (2009) noted that the strict restrictions of the derivation in Stram and Lee (1994) and under the same distributional assumptions, used the less restrictive results of Vu and Zhou (1997) to only obtain the asymptotic distributions of LRTs in LMM with one or two random effects, which are the same as those in Stram and Lee (1994). Crainiceanui and Ruppert (2004), Greven et al. (2008) and Scheipl et al. (2008) studied LMM with crossed designs under normality assumptions where the critical values of LRT and restricted likelihood ratio test (RLRT) also depend on the MC method and the F-tests or the approximate F-tests are also examined which are "the least powerful among the tests under consideration" (Scheipl et al., 2008). In the framework of nonlinear mixed models with clustered designs and under elliptical distribution assumptions

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(Russo et al., 2012), the score-type tests in Silvapulle and Silvapulle (1995) are adapted and the critical values also resort to MC methods. Besides, Zhu and Fung (2004) treated this nonstandard problem as a two-sided problem and a score test was used in the framework of semi-parametric mixed models (SMM) with clustered designs and later, Li and Zhu (2010) proposed a difference-based test T_{mD} which is distribution-free and tractable. Recently, Drikvandi et al. (2013) examined LMM with clustered designs and without distributional assumptions except for moment conditions, but the critical value of the test still depends on the MC method.

Note that (1) and (2) encompass two special cases which are of great interest. One is for clustered designs (Stram and Lee, 1994; Giampaoli and Singer, 2009; Li and Zhu, 2010; Nobre et al., 2013), and the other is for hierarchical and crossed designs (Crainiceanui and Ruppert, 2004; Lin, 1997; Jiang, 2007). For the clustered designs, (1) can be rewritten as

$$Y_i = X_i\beta + Z_i b_i + \epsilon_i, \quad i = 1, 2, \ldots, m \tag{3}$$

where $Y = (Y_1^\tau, \ldots, Y_m^\tau)^\tau, X = [X_1^\tau, X_2^\tau, \ldots, X_m^\tau]^\tau, b = (b_1^\tau, \ldots, b_m^\tau)^\tau$ with b_i being mean zero and the covariance matrix $D_i, Z = \text{diag}(Z_1, \ldots, Z_m)$ and $\epsilon = (\epsilon_1^\tau, \ldots, \epsilon_m^\tau)^\tau$ with $Y_i = (y_{i1}, \ldots, y_{in_i})^\tau, X_i = (X_{i1}, \ldots, X_{in_i})^\tau, Z_i = (Z_{i1}, \ldots, Z_{in_i})^\tau, \epsilon_i = (\epsilon_{i1}, \ldots, \epsilon_{in_i})^\tau$. Thus, $n = \sum_{i=1}^m n_i, D = \text{diag}(D_1, D_2, \ldots, D_m)$. Besides, note that $Y_i's$ are independent for different i and if $D_1 = D_2 = \cdots = D_m$, (3) is the same as that in Laird and Ware (1982).

In the case of hierarchical and crossed designs (Lin, 1997; Jiang, 2007), $Z = [Z_1, Z_2, \ldots, Z_s], b = [\alpha_1^\tau, \alpha_2^\tau, \ldots, \alpha_s^\tau]^\tau$ and then model (1) takes the form

$$Y = X\beta + Z_1\alpha_1 + \cdots + Z_s\alpha_s + \epsilon \tag{4}$$

where $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)^\tau$ and for $k = 1, 2, \ldots, s$, Z_k is the observed $n \times m_k$ matrix associated with the $m_i \times 1$ random effect α_k which has mean zero and the covariance matrix $\sigma_k^2 I_{m_k}$ and $\alpha_1, \ldots, \alpha_s$ are independent. Thus, $D = \text{diag}(\sigma_1^2 I_{m_1}, \sigma_2^2 I_{m_2}, \ldots, \sigma_s^2 I_{m_s})$.

Most literatures are about (3) with clustered designs. For example, Stram and Lee (1994), Giampaoli and Singer (2009), Li and Zhu (2010) and Drikvandi et al. (2013) where a good distribution-free test is introduced for LMMs with longitudinal data but the theoretical theory is unknown and a permutation procedure is adapted to determine the critical value. Besides, Nobre et al. (2013) proposed a tractable U-test T_{mU} for LMM (3) which has the asymptotic normal distribution as that of T_{mD} in Li and Zhu (2010) and the fourth moment of the error is not needed to estimate. Since η_{nrs} is not found there for (4), the specific formula of T_{mU} for (4) is unknown. Li et al. (2014) found that for variance components testing, there are some differences between mixed models with clustered designs and that with crossed designs. Hence, it is of great interest and it becomes the target to develop a tractable test suitable both for LMMs with clustered designs and that with crossed designs without any distribution assumptions except the moment conditions.

The article is organized as follows. Section 2 describes the estimation for some parameters of interest. The residual-based test is developed and its power property is established in Section 3. The numerical analysis is conducted in Section 4 to investigate the performance of the new test and compare it with T_{mD} and other related tests. A concluding remark is in Section 5. Some regular conditions and the detailed proofs, tables and figures are delayed in the Appendix and the supplement.

Before closing this section, we introduce some notations. For any matrix A, $\|A\|^2$ means $A^\tau A$. A^- denotes the generalized inverse matrix of A, and A^+ denotes the Moore–Penrose generalized inverse matrix of A satisfying $AA^+A = A, A^+AA^+ = A^+$, $(AA^+)^\tau = AA^+$ and $(A^+A)^\tau = A^+A$. Besides, let $P_A = A(A^\tau A)^- A$ and $P_{A^\perp} = I - P_A$ where I is the identity matrix and its dimension is the number of rows for A; $rk(A)$ be the rank of the matrix A, and $tr(B)$ denotes the trace of the square matrix B.

2. Estimates

Some estimates are introduced first. By the ordinary least square estimate (LSE) and the transformed weight least square estimate (TWLSE, Li and Zhu, 2010), the estimates of β and σ^2 under the null and the alternative are respectively

$$\hat{\beta}_0 = (X^\tau X)^+ X^\tau Y, \qquad \hat{\beta}_A = (X^\tau P_{Z^\perp} X)^+ X^\tau P_{Z^\perp} Y,$$
$$\hat{\sigma}_0^2 = \frac{Y^\tau P_{X^\perp} Y}{tr(P_{X^\perp})}, \qquad \hat{\sigma}_A^2 = \frac{Y^\tau P_{Z^\perp} P_{\widetilde{X}^\perp} P_{Z^\perp} Y}{tr(P_Z P_{\widetilde{X}^\perp})}, \qquad \widetilde{X} = P_{Z^\perp} X.$$

For model (3) with clustered designs, the fourth moment κ of the error is estimated by

$$\hat{\kappa}_A = 3\hat{\sigma}_A^4 + \frac{\sum_{i=1}^m \left[\left\{ (Y_i - X_i\hat{\beta}_A)^\tau P_{Z_i^\perp} (Y_i - X_i\hat{\beta}_A) \right\}^2 - (n_i - q_i)\hat{\sigma}_A^4 (2 + n_i - q_i) \right]}{\sum_{i=1}^m tr\left\{ \text{diag}^2(P_{Z_i^\perp}) \right\}},$$

which can be derived by the similar method in Li (2011). For model (4), the estimate of κ is estimated as follows

$$\widetilde{\kappa}_A = 3\hat{\sigma}_A^4 + \frac{\sum_{j=1}^n (l_j^\tau Y)^4 - 3\hat{\sigma}_A^4 \left\{ \sum_{j=1}^n (l_j^\tau l_j)^2 \right\}}{\sum_{i=1}^n \sum_{j=1}^n l_{ij}^4} \tag{5}$$

with l_{ij} being the jth element of l_i and l_i being the ith column of the matrix $P_{W^\perp}$ with $W = [X, Z]$.