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Supermodular ordering of Poisson arrays

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ABSTRACT

We derive necessary and sufficient conditions for the supermodular ordering of certain triangular arrays of Poisson random variables, based on the componentwise ordering of their covariance matrices. Applications are proposed for markets driven by jump–diffusion processes, using sums of Gaussian and Poisson random vectors. Our results rely on a new triangular structure for the representation of Poisson random vectors using their Lévy–Khintchine representation.

1. Introduction

Stochastic ordering of random vectors is used in finance and economics as a risk management tool that yields finer information than mean–variance analysis. A d-dimensional random vector $X = (X_1, \ldots, X_d)$ is said to be dominated by another random vector $Y = (Y_1, \ldots, Y_d)$ if

$$E[\Phi(X)] \leqslant E[\Phi(Y)], \qquad (1.1)$$

for all sufficiently integrable $\Phi : \mathbb{R}^d \to \mathbb{R}$ within a certain class of functions that determines the ordering.

One writes $X \leqslant_{\text{sm}} Y$ when (1.1) holds for all *supermodular* functions, i.e. for every function $\Phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that

$$\Phi(x) + \Phi(y) \leq \Phi(x \wedge y) + \Phi(x \vee y), \quad x, y \in \mathbb{R}^d,$$

where the maximum $\vee$ and the minimum $\wedge$ defined with respect to the componentwise order, cf. Meyer and Strulovici (2013) and references therein for a review with economic interpretations.

It is well-known (cf. e.g. Theorem 3.9.5 of Müller and Stoyan (2002)) that necessary conditions for $X \leqslant_{\text{sm}} Y$ are:

(i) X_i and Y_i have same distribution for all $i = 1, \ldots, d$, and
(ii) for all $1 \leq i < j \leq d$ we have

$$\text{Cov}\,(X_i, X_j) \leq \text{Cov}\,(Y_i, Y_j),$$

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where (i) above follows from the fact that any function of a single variable is supermodular, and (ii) follows by application of (1.1) to the supermodular function $\Phi_{i,j}(x_1,\ldots,x_d) := (x_i - E[X_i])(x_j - E[X_j]),\ 1 \le i < j \le d$.

Supermodular ordering of Gaussian random vectors has been completely characterized in Müller and Scarsini (2000) Theorem 4.2, cf. also Theorem 3.13.5 of Müller and Stoyan (2002), by showing that (i) and (ii) above are also sufficient conditions when X and Y are Gaussian, cf. also Müller (2001) for other orderings (stochastic, convex, convex increasing, supermodular) of Gaussian random vectors.

In this paper we consider the supermodular ordering of vectors of Poisson random variables, see Teicher (1954) and Kawamura (1979) for early references on the multivariate Poisson distribution. As noted in Section 2, any d-dimensional Poisson random vector is based on $2^d - 1$ parameters, therefore (i) and (ii), which are based on $d(d+1)/2$ conditions, cannot characterize their distribution ordering except if $d = 2$.

For this reason, in Section 3 we consider a particular dependence structure of Poisson arrays depending on $d(d+1)/2$ parameters based on a natural decomposition of their Lévy measure on the vertices of the d-dimensional unit hypercube. We show in Theorem 4.2 that in this case, conditions (i) and (ii) become necessary and sufficient for the supermodular ordering of X and Y as in the Gaussian setting. When $d = 2$, no restriction has to be imposed on 2-dimensional Poisson random vectors X and Y. Triangular Poisson structures of a different type have been considered in Sim (1993) for the simulation of Poisson random vectors.

Our proof relies on the characterization of the supermodular ordering of d-dimensional infinitely divisible random vectors X and Y by their Lévy measures, cf. Bäuerle et al. (2008), based on the covariance identities of Houdré et al. (1998). Extensions of such identities have already been applied to other multidimensional stochastic (including convex) orderings in e.g. Bergenthum and Rüschendorf (2007) based on stochastic calculus and in Arnaudon et al. (2008) using forward–backward stochastic calculus.

We also derive sufficient conditions for the supermodular ordering of sums of Gaussian and Poisson random vectors, with application to a jump–diffusion asset price model, cf. Theorem 4.4. Indeed, the supermodular ordering of random asset vectors implies the stop-loss ordering of their associated portfolios, cf. Theorem 3.1 in Müller (1997) or Theorem 8.3.3 in Müller and Stoyan (2002).

We proceed as follows. In Section 2 we recall the construction of Poisson random vectors, and in Section 3 we specialize this construction to a certain dependence setting based on Poisson arrays. Finally in Section 4 we prove our main characterization of supermodular ordering for such vectors, including extensions to the increasing supermodular order, cf. Theorem 4.2 and Proposition 4.3. In addition we provide a sufficient condition for the supermodular ordering of sums of Gaussian and Poisson random vectors in Theorem 4.4, with application to an exponential jump–diffusion asset price model. We also include a remark on the related convex ordering problem for such Poisson arrays in Proposition 4.5.

2. Poisson random vectors

Consider $X = (X_i)_{1\leqslant i\leqslant d}$ a d-dimensional infinitely divisible Poisson random vector with Lévy measure μ on $\mathbb{R}^d$, which satisfies

$$E[e^{i\langle \bar{t},X\rangle}] = \exp\left(\int_{\mathbb{R}^d} (e^{i\langle \bar{t},x\rangle} - 1)\mu(dx)\right),$$

where $\bar{t} = (t_1,\ldots,t_d) \in \mathbb{R}^d$ and $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^d$.

Since $(X_i)_{1\leqslant i\leqslant d}$ has Poisson marginals, all marginals of μ on $\mathbb{R}^d$ are supported by $\{0, 1\}$ and consequently the Lévy measure $\mu(dx)$ is supported on the vertices of the unit hypercube of $\mathbb{R}^d$ and takes the form

$$\mu(dx) = \sum_{\emptyset\neq S\subset\{1,2,\ldots,d\}} a_S\delta_{e_S}(dx),$$

where $(a_S)_{\emptyset\neq S\subset\{1,2,\ldots,d\}}$ is a family of nonnegative numbers and

$$C_d = \{0, 1\}^d = \left\{ e_S := \sum_{i\in S} e_i : S \subset \{1,\ldots,d\}\right\}$$

denotes the vertices of the d-dimensional unit hypercube, identified to the power set $\{0, 1\}^d \simeq \{S \in \{1,\ldots,d\}\}$ of $\{1,\ldots,d\}$, and $(e_k)_{1\leqslant k\leqslant d}$ is the canonical basis of $\mathbb{R}^d$.

Consequently, any d-dimensional Poisson random vector $X = (X_1,\ldots,X_d)$ can be represented as

$$X_i = \sum_{\substack{S\in\{0,1\}^d\\ S\neq 0}} \mathbf{1}_{\{i\in S\}}X_S = \sum_{\substack{S\subset\{1,2,\ldots,d\}\\ S\ni i}} X_S, \quad i = 1,\ldots,d, \tag{2.1}$$

where $(X_S)_{\emptyset\neq S\subset\{1,2,\ldots,d\}}$ is a family of $2^d - 1$ independent Poisson random variables with respective intensities $(a_S)_{\emptyset\neq S\subset\{1,2,\ldots,d\}}$, cf. also Theorem 3 of Kawamura (1979).