



On degenerate stochastic equations of Itô type with jumps

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ABSTRACT

The one-dimensional stochastic equation $dX_t = b_1(X_t)dW_t + b_2(X_{t-})dZ_t + a(X_t)dt$, $t \geq 0$, where $b_1, b_2, a : \mathbb{R} \rightarrow \mathbb{R}$ are Borel measurable functions, W is a Brownian motion, and Z is a symmetric stable process of index $0 < \alpha < 2$, is considered. We prove the existence of (weak) solutions under some conditions of boundedness of coefficients when b_1 can be degenerate which improves the results of Lepeltier and Marchal [Lepeltier, J.P., Marchal, B., 1976. Problème des martingales et équations différentielles stochastiques associées à un opérateur intégral-différentiel. Ann. IHP 12 (1), 43–103] for this case. Our approach is based on Krylov's estimates for solutions X and the weak convergence arguments.

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1. Introduction

In recent years one can observe an increased interest, from the side of praxis as well as from theory, in stochastic models allowing jumps. In particular, the so-called diffusion models with jumps appear to attract high attention. The purpose of this note is to consider the following model

$$dX_t = b_1(X_t)dW_t + b_2(X_{t-})dZ_t + a(X_t)dt, \quad t \geq 0, \quad (1.1)$$

where $b_1, b_2, a : \mathbb{R} \rightarrow \mathbb{R}$ are measurable coefficients, W is a Brownian motion, and Z is a symmetric stable process of index $0 < \alpha < 2$. In this model the first term on the right-hand side of (1.1) describes the continuous fluctuations, the second term reflects the jump behaviour, and the third summand represents the drift in the dynamics of the resulting process X .

The goal is to construct a (weak) solution of Eq. (1.1) with only measurable coefficients when the leading diffusion coefficient b_1 is allowed to be *degenerate*. Stochastic models with measurable coefficients are important in praxis, in particular, in control theory. The models of the form (1.1) without jumps and with only measurable coefficients were first considered by Krylov (1980) and the method used by him was the one based on the corresponding integral estimates for solutions X satisfying the SDEs (1.1). On the other hand, those integral estimates were derived using some important analytical inequalities for generators of diffusion processes X . Krylov's estimates turned out to be an important and useful tool in the theory of stochastic processes.

Various authors tried to generalize Krylov's estimates for some other classes of processes X different from diffusion ones. Thus, Melnikov (1983) derived the corresponding estimates for processes X being continuous semimartingales. Anulova and Pragarauskas (1977) obtained Krylov's estimates for a class of diffusion processes with jumps generated by a Cauchy process. One should notice that in the mentioned generalizations one relied essentially on the use of the original Krylov's inequalities for the generators of diffusion processes. Here we prove the corresponding Krylov's estimates directly using the Fourier transforms.

An alternative approach to study the existence of weak solutions of Eq. (1.1) is to consider a corresponding *martingale problem* associated with Eq. (1.1). Such an approach was used by Lepeltier and Marchal (1976) where they studied an even

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more general class of stochastic equations with jumps and a continuous Brownian motion component. The main novelty of our results compared with those of Lepeltier and Marchal (1976) in terms of the existence of solutions of Eq. (1.1) is that we prove the existence of solutions in a degenerate case whenever Lepeltier and Marchal (1976) assumed the coefficient b_1 to be uniformly bounded away from zero.

The note is organized as follows. For the convenience of the reader, in Section 2 we give some definitions and collect a few known facts to be used later. Section 3 deals with derivation of some important analytical and stochastic estimates similar to those derived originally by Krylov for the diffusion case. Using the obtained estimates, we prove in Section 4 the existence of solutions for Eq. (1.1).

2. Preliminaries

We shall denote by $\mathbf{D}_{[0,\infty)}(\mathbb{R})$ the Skorokhod space, i.e. the set of all real-valued functions $z : [0, \infty) \rightarrow \mathbb{R}$ with right-continuous trajectories and with finite left limits (also called *càdlàg* functions). For simplicity, we shall write $\mathbf{D}$ instead of $\mathbf{D}_{[0,\infty)}(\mathbb{R})$. We will equip $\mathbf{D}$ with the σ -algebra $\mathcal{D}$ generated by the Skorokhod topology. Under $\mathbf{D}^n$ we will understand the n -dimensional Skorokhod space defined as $\mathbf{D}^n = \mathbf{D} \times \cdots \times \mathbf{D}$ with the corresponding σ -algebra $\mathcal{D}^n$ being the direct product of n one-dimensional σ -algebras $\mathcal{D}$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space carrying a process Z with $Z_0 = 0$ and let $\mathbb{F} = (\mathcal{F}_t)$ be a filtration on $(\Omega, \mathcal{F}, \mathbf{P})$. The notation $(Z, \mathbb{F})$ means that Z is adapted to the filtration $\mathbb{F}$. We call $(Z, \mathbb{F})$ a symmetric stable process of index $\alpha \in (0, 2]$ if trajectories of Z belong to $\mathbf{D}$ and

$$\mathbf{E}(e^{i\xi(Z_t - Z_s)} | \mathcal{F}_s) = e^{-(t-s)|\xi|^\alpha}$$

for all $t > s \geq 0$ and $\xi \in \mathbb{R}$. If $\alpha = 2$, then $Z = W$ is a Brownian motion with the variance $2t$. It is well known that Brownian motion W is the only symmetric stable process with continuous paths.

For all $\alpha \in (0, 2]$, the process Z is a semimartingale as a Levy process and the stochastic integral $\int_0^t f_s dZ_s$ can be defined via the general semimartingale approach (cf. Chapter 2 in Protter (2004)). From another side, the given stochastic integral can also be defined as the Itô stochastic integral using some isometry properties between the corresponding spaces of integrators and integrals. The latter construction is due to K. Itô who originally applied it to define the integral in the case of $\alpha = 2$ (cf. Chapter 2 in Ikeda and Watanabe (1989)). For the case of an arbitrary $\alpha \in (0, 2]$, Itô's approach was generalized by Rosiński and Woyczyński (1986).

If $\alpha \in (0, 2)$ one has the following quasi-isometrical property: there exist constants c_α and C_α depending on α only such that for all $t > 0$

$$c_\alpha \mathbf{E} \int_0^t |f_s|^\alpha ds \leq \sup_{\lambda > 0} \lambda^\alpha \mathbf{P} \left(\sup_{s \leq t} \left| \int_0^s f_u dZ_u \right| > \lambda \right) \leq C_\alpha \mathbf{E} \int_0^t |f_s|^\alpha ds. \quad (2.1)$$

The formula (2.1) was proven by Rosiński and Woyczyński (1986, Theorem 2.1), and generalizes the classical Itô isometry in the Brownian motion case: $\mathbf{E}[\int_0^t f_s dW_s]^2 = \mathbf{E} \int_0^t f_s^2 ds$. Moreover, (2.1) is used to prove that the stochastic integral $\int_0^t f_s dZ_s$ exists if and only if $\int_0^t |f_s|^\alpha ds < \infty$ a.e.

For all $0 < \alpha \leq 2$, Z is a Markov process and can be characterized in terms of its infinitesimal generator $\mathcal{L}$. Thus for $\alpha < 2$

$$(\mathcal{L}g)(x) = \int_{\mathbb{R} \setminus \{0\}} [g(x+z) - g(x) - \mathbf{1}_{\{|z|<1\}} g'(x)z] \frac{k_\alpha}{|z|^{1+\alpha}} dz$$

for any $g \in C^2$, where C^2 is the set of all bounded and twice continuously differentiable functions $g : \mathbb{R} \rightarrow \mathbb{R}$ and k_α is a suitable constant. Contrary to the case of $\alpha \in (0, 2)$, the infinitesimal generator of Brownian motion ($\alpha = 2$) is the second derivative operator.

We notice also that the use of Fourier transform can simplify calculations when working with infinitesimal generator $\mathcal{L}$. Let $g \in L_1(\mathbb{R})$ and

$$\hat{g}(\xi) := \int_{\mathbb{R}} e^{i\xi z} g(z) dz$$

be the Fourier transform of g . The following facts are well known so that we omit their proof.

Proposition 2.1. (i) Assume that $g \in C^2(\mathbb{R})$ and $\mathcal{L}g \in L_1(\mathbb{R})$. Then

$$\widehat{(\mathcal{L}g)}(\xi) = -|\xi|^\alpha \hat{g}(\xi).$$

(ii) For any $k = 1, 2, \dots$, let $g^{(k-1)}$ be absolutely continuous on every compact subset of $\mathbb{R}$ and $g^{(k)} \in L_1(\mathbb{R})$. Then

$$\widehat{g^{(k)}}(\xi) = (-i\xi)^k \hat{g}(\xi).$$

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