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On moments of recurrence times for positive recurrent renewal sequences

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ABSTRACT

The explicit formula for the moments of recurrence times for positive recurrent renewal sequences is established.

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1. Introduction and result

Let $\{X_k\}_{k\in\mathbb{Z}_+}$ be a recurrent Markov chain with state space $\mathbb{N}\cup\{0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Denote by T_m the time of mth return to zero (see Breiman (1968), p. 122) and put $f_n = \mathrm{P}[T_1 = n]$, $n \geq 1$, $u_n = \mathrm{P}[X_n = 0 | X_0 = 0]$, $n \geq 0$. Thus the sequence u_n satisfies

$$u_n = \sum_{k=1}^{n} f_k u_{n-k} = \sum_{k=0}^{n-1} u_k f_{n-k}, \quad n \geq 1,\ u_0 = 1 \tag{1.1}$$

and

$$\sum_{k=1}^{\infty} f_k = 1, \quad f_k \geq 0,\ f_0 = 0. \tag{1.2}$$

Conversely, for a recurrent renewal sequence u_n, i.e. satisfying (1.1) and (1.2), there exists a recurrent Markov chain with $f_k = \mathrm{P}[T_1 = k]$ (see Orey (1971), Chapter 3). It is well known that recurrence times T_m, $m \in \mathbb{N}$, form an i.i.d. sequence (see Breiman (1968), p. 139). So the limit theory applies when we are able to calculate the moments $m_k = \mathrm{E}[T_1^k]$. For a positive recurrent renewal sequence u_n, i.e. satisfying $m_1 < \infty$, it follows from the Karamata tauberian theorem that

$$m_1 = \lim_n \frac{n}{\sum_{k=0}^{n} u_k}.$$

The case $k = 2$ is less evident

$$m_2 = m_1 + 2m_1^2 \sum_{n\geq 0}\left(u_n - \frac{1}{m_1}\right)$$

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(see Feller (1970), XIII, Sections 12, 19 and Chung (1967), p. 33). To the author's best knowledge, similar formulae for higher moments are unknown. This paper aims to fill this gap.

Define the generating functions

$$F(s) = \sum_{n=1}^{\infty} f_n s^n, \qquad B(s) = \sum_{n=0}^{\infty} \left(u_n - \frac{1}{m_1} \right) s^n.$$

The main result is the following statement.

Theorem 1. *For $k \geq 0$*

$$F_{k+1} = F_1 \sum_{\substack{(n_1, n_2, \ldots, n_k) \\ n_i \in \mathbb{Z}_+, \sum_{i=1}^{k} i n_i = k}} \frac{(n_1 + n_2 + \cdots + n_k)!}{n_1! n_2! \cdots n_k!} (F_1 B_0)^{n_1} (F_1 B_1)^{n_2} \cdots (F_1 B_{k-1})^{n_k}$$

where

$$F_k = \frac{1}{k!} \frac{\partial^k}{\partial s^k} F(s) \bigg|_{s=1}, \qquad B_k = \frac{1}{k!} \frac{\partial^k}{\partial s^k} B(s) \bigg|_{s=1}.$$

Now, denoting by $\left\{ {k \atop \nu} \right\}$ the Stirling numbers of the second kind one can easily obtain the required formula

$$m_k = \sum_{n=0}^{\infty} n^k f_n = \sum_{\nu=0}^{k} \left\{ {k \atop \nu} \right\} \nu! F_\nu, \quad k \geq 1. \tag{1.3}$$

In particular, taking $k = 3$ in (1.3) we obtain by Theorem 1

$$\begin{aligned} m_3 &= \left\{ {3 \atop 0} \right\} F_0 + \left\{ {3 \atop 1} \right\} F_1 + \left\{ {3 \atop 2} \right\} 2! F_2 + \left\{ {3 \atop 3} \right\} 3! F_3 \\ &= 0 \cdot F_0 + 1 \cdot F_1 + 3 \cdot 2 \cdot F_1^2 B_0 + 1 \cdot 6 \cdot (F_1^3 B_0^2 + F_1^2 B_1) \\ &= m_1 + 6 m_1^2 \sum_{n=0}^{\infty} \left(u_n - \frac{1}{m_1} \right) + 6 m_1^3 \left(\sum_{n=0}^{\infty} \left(u_n - \frac{1}{m_1} \right) \right)^2 + 6 m_1^2 \sum_{n=0}^{\infty} n \left(u_n - \frac{1}{m_1} \right). \end{aligned}$$

2. Proofs

Let $\{a_n\}_{n \in \mathbb{Z}_+}$ be a sequence of numbers. For $k \geq 1$ define

$$a_n^{(k)} = \sum_{\nu=n+1}^{\infty} a_\nu^{(k-1)}, \qquad A_k(s) = \sum_{n \geq 0} a_n^{(k)} s^n, \qquad A_k = A_k(1)$$

where $a_n^{(0)} = a_n$. Set $A_0(s) = \sum_{n \geq 0} a_n s^n, A_0 = A_0(1)$.

Lemma 1.

$$(1 - s) A_k(s) = A_{k-1}(1) - A_{k-1}(s), \quad k \geq 1.$$

Proof of Lemma 1. By the definition of $A_k(s)$ we have

$$\begin{aligned} (1 - s) A_k(s) &= (1 - s) \sum_{n \geq 0} a_n^{(k)} s^n = (1 - s) \sum_{n \geq 0} \left(\sum_{\nu=n+1}^{\infty} a_\nu^{(k-1)} \right) s^n \\ &= (1 - s) \sum_{n \geq 0} \left(\sum_{\nu=n}^{\infty} a_{\nu+1}^{(k-1)} \right) s^n = (1 - s) \sum_{\nu=0}^{\infty} a_{\nu+1}^{(k-1)} \sum_{n=0}^{\nu} s^n \\ &= \sum_{\nu=1}^{\infty} a_\nu^{(k-1)} + a_0^{(k-1)} - \sum_{\nu=1}^{\infty} a_\nu^{(k-1)} s^\nu - a_0^{(k-1)} s^0 \\ &= A_{k-1}(1) - A_{k-1}(s). \end{aligned}$$

This proves the lemma. □

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