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Precise asymptotics for a new kind of complete moment convergence☆

Weidong Liu*, Zhengyan Lin

Department of Mathematics, Zhejiang University, Hangzhou 310027, China

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Abstract

In this paper, we introduce a new kind of complete moment convergence, which includes complete convergence as a special case. And we also achieves some results about precise asymptotics for this kind of complete moment convergence. For example, let $\{X, X_n; n\geqslant 1\}$ be a sequence of i.i.d. random variables, then

$$\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\sum_{n=1}^{\infty}\frac{1}{n^2}\mathsf{E}S_n^2 I\{|S_n|\geqslant\varepsilon n\} = 2\sigma^2$$

holds, if and only if $\mathsf{E}X=0$, $\mathsf{E}X^2=\sigma^2$ and $\mathsf{E}X^2\log^+|X|<\infty$.

MSC: Primary 60F15; secondary 60G50

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1. Introduction and main result

Throughout this paper, we let $\{X, X_n; n\geqslant 1\}$ be a sequence of i.i.d. random variables, and set $S_n=\sum_{k=1}^n X_k$. Throughout, C shall denote positive constants, possibly varying from place to place, and $[x]$ shall denote the largest integer $\leqslant x$.

Since Hsu and Robbins (1947) introduced the concept of complete convergence, there have been extensions in several directions. One of them is to discuss the precise rate. Heyde (1975) proved that

$$\lim_{\varepsilon\searrow 0}\varepsilon^2\sum_{n=1}^{\infty}\mathsf{P}(|S_n|\geqslant\varepsilon n)=\mathsf{E}X^2, \tag{1.1}$$

when $\mathsf{E}X=0$, and $\mathsf{E}X^2<\infty$. For analogous results in the more general case, see Chen (1978), Gut and Spătaru (2000), Spătaru (1999), etc.

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*Corresponding author.
E-mail addresses: Liuwd99@tom.com (W. Liu), zlin@zju.edu.cn (Z. Lin).

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By Markov's inequality, we have

$$T_{p,\alpha}(\varepsilon) := \sum_{n=1}^{\infty} n^{p\alpha-2}\mathsf{P}(|S_n| \geqslant \varepsilon n^{\alpha}) \leqslant \varepsilon^{-q} \sum_{n=1}^{\infty} n^{p\alpha-2-\alpha q}\mathsf{E}|S_n|^q I\{|S_n| \geqslant \varepsilon n^{\alpha}\},$$

where $0 \leqslant q \leqslant p$ and $\varepsilon > 0$. Baum and Katz (1965) discussed the convergence of $T_{p,\alpha}(\varepsilon)$. Chow (1988) discussed the convergence of

$$B_{p,\alpha}(\varepsilon) := \sum_{n=1}^{\infty} n^{p\alpha-\alpha-2}\mathsf{E}\{|S_n| - \varepsilon n^{\alpha}\}_+, \quad \varepsilon > 0.$$

Set

$$G_{q,p,\alpha}(\varepsilon) = \sum_{n=1}^{\infty} n^{p\alpha-2-\alpha q}\mathsf{E}|S_n|^q I\{|S_n| \geqslant \varepsilon n^{\alpha}\}.$$

We can see that $T_{p,\alpha}(\varepsilon) = G_{0,p,\alpha}(\varepsilon)$, and $B_{p,\alpha}(\varepsilon) = G_{1,p,\alpha}(\varepsilon) - \varepsilon G_{0,p,\alpha}(\varepsilon)$. So, it is worth to study the properties of $G_{q,p,\alpha}(\varepsilon)$. In this paper, we give the precise rates of convergence of $G_{q,p,\alpha}(\varepsilon)$ when $\varepsilon \searrow 0$. For the sake of convenience, we discuss the case $\alpha = 1$, $p = 2$, and $0 \leqslant q \leqslant 2$. Our results are stated as follows.

Theorem 1. *Suppose that* $\{X, X_n; n \geqslant 1\}$ *is a sequence of i.i.d. random variables,*

$$\mathsf{E}X = 0, \quad \mathsf{E}X^2 = \sigma^2 \quad \textit{and} \quad \mathsf{E}X^2 \log^+ |X| < \infty. \tag{1.2}$$

Then we have

$$\lim_{\varepsilon \searrow 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} \mathsf{E}S_n^2 I\{|S_n| \geqslant \varepsilon n\} = 2\sigma^2. \tag{1.3}$$

Conversely, if (1.3) *is true, then* (1.2) *holds.*

Theorem 2. *Suppose that* $\{X, X_n; n \geqslant 1\}$ *is a sequence of i.i.d. random variables,*

$$\mathsf{E}X = 0, \quad \mathsf{E}X^2 = \sigma^2 < \infty. \tag{1.4}$$

Then we have

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2-p} \sum_{n=1}^{\infty} \frac{1}{n^p} \mathsf{E}|S_n|^p I\{|S_n| \geqslant \varepsilon n\} = \frac{2}{2-p}\sigma^2, \tag{1.5}$$

for $0 \leqslant p < 2$. *Conversely, if* (1.5) *is true for some* $0 \leqslant p < 2$, *then* (1.4) *holds.*

Davis (1968) proved that

$$\sum_{n=1}^{\infty} \frac{\log n}{n} \mathsf{P}(|S_n| \geqslant \varepsilon \sqrt{n \log n}) < \infty, \quad \varepsilon > 0, \tag{1.6}$$

if and only if $\mathsf{E}X = 0$ and $\mathsf{E}X^2 < \infty$. The natural analog to Theorem 1, corresponding to (1.6), runs as follows.

Theorem 3. *Suppose that* $\{X, X_n; n \geqslant 1\}$ *is a sequence of i.i.d. random variables,* $0 < \delta \leqslant 1$,

$$\mathsf{E}X = 0, \quad \mathsf{E}X^2 = \sigma^2 \quad \textit{and} \quad \mathsf{E}X^2 (\log^+ |X|)^{\delta} < \infty. \tag{1.7}$$

Then we have

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2\delta} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} \mathsf{E}S_n^2 I\{|S_n| \geqslant \varepsilon \sqrt{n \log n}\} = \frac{\sigma^{2\delta+2}}{\delta} \mathsf{E}|N|^{2\delta+2}. \tag{1.8}$$

Conversely, if (1.8) *is true, then* (1.7) *holds.*