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On Lévy measures for infinitely divisible natural exponential families

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Abstract

We link the infinitely divisible measure μ to its modified Lévy measure $\rho = \rho(\mu)$ in terms of their variance functions, where $x^{-2}[\rho(\mathrm{d}x) - \rho(\{0\})\delta_0(\mathrm{d}x)]$ is the Lévy measure associated with μ. We deduce that, if the variance function of μ is a polynomial of degree $p \geqslant 2$, then, the variance function of ρ is still a second degree polynomial. We illustrate these results with some Lévy processes such as positive stable and a class of Poisson stopped-sum processes.

MSC: primary 60E07; secondary 60G51; 62E10

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1. Introduction

Infinitely divisible probability measures are by definition probability measures whose nth roots (in the convolution sense) exist for any positive integer n. They are closely related to the Lévy processes (e.g. Bertoin, 1996; Sato, 1999). We also recall that a Lévy process $\mathscr{X} = \{X_t;\ t \geqslant 0\} \equiv \mathscr{X}_\mu$ governed by an infinitely divisible law μ is a càdlàg stochastic process with stationary and independent increments and $X_0 = 0$.

In order to study the behaviour of a Lévy process $\mathscr{X} = \mathscr{X}_\mu$ governed by μ, the well-known Lévy–Khintchine characterization is frequently used: a probability measure μ on $\mathbb{R}$ is infinitely divisible if and only if there exist $\gamma, \sigma \in \mathbb{R}$ and a positive finite measure v on $\mathbb{R}\backslash\{0\}$ satisfying

$$\int_{\mathbb{R}\backslash\{0\}} \min(1, x^2) v(\mathrm{d}x) < \infty \tag{1}$$

such that the Fourier transform of μ is of the form

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\theta x} \mu(\mathrm{d}x) = \exp\left\{\mathrm{i}\gamma\theta - \frac{\sigma^2\theta^2}{2} + \int_{\mathbb{R}\backslash\{0\}} [\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta\tau(x)] v(\mathrm{d}x)\right\}, \tag{2}$$

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where τ is some fixed bounded continuous function on $\mathbb{R}$ such that $(\tau(x)-x)/x^2$ is bounded as $x \to 0$ (e.g. $\tau(x) = x/(1+x^2)$ or $\tau(x) = \sin x$). In this case the triple (γ, σ^2, v) is unique, and the measure $v = v(\mu)$ satisfying (1) is called the *Lévy measure* of μ. From (2), the first characteristic γ is connected to the drift of the process $\mathscr{X}_\mu$, whereas σ^2 is the infinitesimal variance of the Brownian motion part of $\mathscr{X}_\mu$, and v determines the probabilistic character of the jumps of $\mathscr{X}_\mu$. For example, a compound Poisson process is a Lévy process $\mathscr{X}$ with $\gamma = \sigma = 0$ and v is a finite measure. When v is not a finite measure, $\mathscr{X}$ is not a compound Poisson process; so $\mathscr{X}$ has infinitely many jumps in every finite time interval of strictly positive length. Consequently, one may define infinitely divisible distributions in terms of their Lévy measure. Recall here that the Lévy measure $v = v(\mu)$ is classified from (1) in three types as follows: if v is bounded then it is said to be of type 0. If v is unbounded and $\int_{\mathbb{R}\setminus\{0\}} \min(1,|x|)v(\mathrm{d}x) < \infty$ it is said to be of type 1. It is said to be of type 2 if $\int_{\mathbb{R}\setminus\{0\}} \min(1,|x|)v(\mathrm{d}x)$ diverges.

Another way to determine the Lévy measure is to consider the remarkable Letac (1992, p. 12) characterization which we recall in the following proposition; see also Seshadri (1993, Theorem 5.3). Let $\mathscr{M}$ denote the set of σ-finite positive measures μ on $\mathbb{R}$ not concentrated at one point, with the Laplace transform of μ given by

$$L_\mu(\theta) = \int_{\mathbb{R}} \exp(\theta x)\mu(\mathrm{d}x)$$

and such that the interior $\Theta(\mu)$ of the interval $\{\theta \in \mathbb{R} : L_\mu(\theta) < \infty\}$ is non-empty.

Proposition 1. *Let $\mu \in \mathscr{M}$. Then, μ is infinitely divisible if and only if there exists a positive measure $\rho = \rho(\mu)$ such that for all $\theta \in \Theta(\mu)$*

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \log L_\mu(\theta) = L_\rho(\theta). \tag{3}$$

In this case $\Theta(\rho) = \Theta(\mu)$, and if $\theta_0 \in \Theta(\mu)$ then the Lévy measure v_{θ_0} corresponding to the probability $P(\theta_0, \mu) = \mathrm{e}^{\theta_0 x}\mu(\mathrm{d}x)/\int_{\mathbb{R}} \mathrm{e}^{\theta_0 x}\mu(\mathrm{d}x)$ is

$$v_{\theta_0}(\mathrm{d}x) = x^{-2}\exp\{\theta_0 x\}[\rho(\mathrm{d}x) - \rho(\{0\})\delta_0(\mathrm{d}x)], \tag{4}$$

where δ_0 denotes the Dirac mass at 0.

The measure $\rho = \rho(\mu)$ described in Proposition 1 is called the *modified Lévy measure* of μ. A natural question that arises is related to the properties of the associated measure ρ. The aim of this paper is to provide the properties of the natural exponential family (NEF) $G = G(\rho)$ generated by ρ. Theorem 2 along with Corollary 3 give an answer to such a question in Section 2. In particular, they present the quadratic behaviour of the NEF G under certain condition of the NEF $F = F(\mu)$ generated by μ. Finally in Section 3, we present two interesting examples to illustrate these properties.

2. Main results

Let us collect some definition which will be used in the sequel (for more details cf. for instance, Kotz et al., 2000, Chapter 54).

Let $K_\mu(\theta) = \log L_\mu(\theta)$ be the cumulant function. The NEF $F = F(\mu)$ generated by $\mu \in \mathscr{M}$ is the family of probability measures $\{P_{\theta,\mu}(\mathrm{d}x) = \exp[\theta x - K_\mu(\theta)]\mu(\mathrm{d}x); \theta \in \Theta(\mu)\}$. If X is a random variable distributed according to $P_{\theta,\mu}$, then $\mathbb{E}_\theta(X) = K'_\mu(\theta)$ and $\mathrm{var}_\theta(X) = K''_\mu(\theta)$. The function $m(\theta) = K'_\mu(\theta)$ is a one-to-one transformation from $\Theta(\mu)$ onto $M_F = K'_\mu(\Theta(\mu))$ and thus $m = m(\theta)$ provides an alternative parametrization of the family $F = \{P(m,F); m \in M_F\}$, called the mean parametrization. Note that M_F depends only on F, and not on the choice of the generating measure μ of F. The variance of $P(m,F)$ can be written as a function of the mean parameter m, $V_F(m) = K''_\mu(\theta)$, called the *variance function* of F. Together with the mean domain M_F, V_F characterizes F within the class of all NEFs. This led Morris (1982) to establish the first classification of NEFs with quadratic variancefunctions on $\mathbb{R}$, whose class contains six basic families, as gamma and negative binomial, up to affine transformation and convolution power. For many common distributions in $\mathbb{R}$, V_F presents an expression simpler than the density of $P(m,F)$ or its Laplace transform (e.g. Bar-Lev, 1987; Letac and Mora, 1990; Bar-Lev et al., 1992, Jørgensen, 1997; Kokonendji et al., 2004). Finally, we note that if μ is a