Statistics and Probability Letters 88 (2014) 66–72

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

On perturbation bounds for continuous-time Markov chains

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ARTICLE INFO

Article history:
Received 25 October 2013
Received in revised form 27 January 2014
Accepted 31 January 2014
Available online 6 February 2014

MSC:
60J271
60J22
60J28

Keywords:
Continuous-time Markov chains
Inhomogeneous Markov chains
Perturbation bounds
Special norms

ABSTRACT

We suggest an approach to obtaining general estimates of stability in terms of special "weighted" norms related to total variation. Two important classes of continuous-time Markov chains are considered for which it is possible to obtain exact convergence rate estimates (and hence, guarantee exact stability estimates): birth–death–catastrophes processes, and queueing models with batch arrivals and group services.

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1. Introduction

The problems related to the estimation of stability of Markov chains with respect to perturbations of their characteristics in various cases and for various classes of processes have been thoroughly studied starting from the pioneering papers of Kartashov (1985, 1986) and (in parallel) Zeifman (1985) published in 1980s.

To obtain efficient and exact estimates for perturbations it is required to obtain estimates of the convergence rate of Markov chains to their limit characteristics in the form of explicit inequalities; moreover, the more accurate convergence rate estimates are, the better stability estimates can be obtained.

These estimates are most easily obtained for finite stationary (homogeneous) chains; therefore, it is the case most results deal with. See, e.g., Aldous and Fill (2002), Diaconis and Stroock (1991), Mitrophanov (2003, 2004).

In the case of countable (and even more, inhomogeneous) chains very serious problems arise related, mainly, to the absence of uniform ergodicity for processes most interesting from the point of view of practical applications. For example, birth–death processes modeling many queueing, biological, chemical, physical processes are *not* uniformly ergodic under natural assumptions.

Following the original ideas considered in detail in Kartashov (1996), most papers use probability methods to study stationary chains (with finite, countable or general phase space) and their ergodicity and stability in the original norm and

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http://dx.doi.org/10.1016/j.spl.2014.01.031
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related "weighted" norms yielding uniform ergodicity for a wider class of chains; see a review of the results obtained in this direction as well as recent achievements were presented in Liu (2012).

In our papers the same problems for inhomogeneous finite or countable continuous-time chains were studied by another technique.

A general approach to studying the stability of the vector of state probabilities of a continuous-time Markov chain in the total variation norm under the perturbation of the infinitesimal characteristics of the chain was first considered in Zeifman (1985); also see Zeifman (1991), Zeifman and Isaacson (1994). In Zeifman (1998) the estimates of stability under relatively small perturbations were studied in detail for strongly inhomogeneous birth–death processes. The problems of stability and the corresponding estimates were considered for new classes of processes in Zeifman and Korotysheva (2012), Zeifman et al. (2014).

In the present paper we propose an approach to the construction of general estimates for the perturbation bounds of Markov chains in terms of special "weighted" norms related to total variation.

Section 2 contains preliminary material. The general result is presented is Section 3. In Section 4 two important examples of classes of continuous-time Markov chains are considered for which it is possible to obtain exact convergence rate estimates, and hence, guarantee exact perturbation bounds.

2. Preliminaries

Let $X = X(t), t \geq 0$ be an inhomogeneous, in general, continuous-time Markov chain. Let $p_{ij}(s,t) = \Pr\{X(t) = j|X(s) = i\}, i, j \geq 0,\ 0 \leq s \leq t$, be the transition probabilities for $X = X(t)$, $p_i(t) = \Pr\{X(t) = i\}$ be its state probabilities, and $\mathbf{p}(t) = (p_0(t), p_1(t), \ldots)^T$ be the corresponding probability distribution. Throughout the paper we assume that

$$\Pr\{X(t+h) = j|X(t) = i\} = \begin{cases} q_{ij}(t)\,h + \alpha_{ij}(t,h), & \text{if } j \neq i, \\ 1 - \sum\limits_{k\neq i} q_{ik}(t)\,h + \alpha_i(t,h), & \text{if } j = i, \end{cases} \quad (1)$$

where $q_{ij}(t)$ denotes the intensity of transition from state i to state j, $\alpha_{ij}(t,h) = o(h)$ for any t as $h \to 0$. Moreover, we suppose that all $\alpha_i(t,h)$ are $o(h)$ uniform in i, i.e., $\sup_i |\alpha_i(t,h)| = o(h)$.

We assume that (in the inhomogeneous case) all intensity functions are linear combinations of a finite number of nonnegative functions which are locally integrable on $[0,\infty)$.

Put $a_{ij}(t) = q_{ji}(t)$ for $j \neq i$ and let $a_{ii}(t) = -\sum_{j\neq i} a_{ji}(t) = -\sum_{j\neq i} q_{ij}(t)$. In addition, applying our standard approach (see details in Granovsky and Zeifman (2004), Zeifman et al. (2006)) we suppose that the intensity matrix is essentially bounded, i.e.,

$$|a_{ii}(t)| \leq L < \infty, \quad (2)$$

for almost all $t \geq 0$. Then the probabilistic dynamics of the process is represented by the forward Kolmogorov system:

$$\frac{d\mathbf{p}}{dt} = A(t)\mathbf{p}(t), \quad (3)$$

where $A(t)$ is the transposed intensity matrix of the process.

Throughout the paper by $\|\cdot\|$ we denote the l_1-norm, i.e., $\|\mathbf{x}\| = \sum |x_i|$, and $\|B\| = \sup_j \sum_i |b_{ij}|$ for $B = (b_{ij})_{i,j=0}^{\infty}$. Let Ω be the set all stochastic vectors, i.e., l_1-vectors with nonnegative coordinates and unit norm. Then we have $\|A(t)\| = 2\sup_k |a_{kk}(t)| \leq 2L$ for almost all $t \geq 0$. Hence, the operator function $A(t)$ from l_1 into itself is bounded for almost all $t \geq 0$ and locally integrable on $[0;\infty)$. Therefore, we can consider (3) as a differential equation in the space l_1 with the bounded operator.

It is well known (see Daleckij and Krein (1974)) that the Cauchy problem for the differential equation (3) has a unique solution for an arbitrary initial condition, and $\mathbf{p}(s) \in \Omega$ implies $\mathbf{p}(t) \in \Omega$ for $t \geq s \geq 0$. Hence, we can put $p_0(t) = 1 - \sum_{i\geq 1} p_i(t)$ and obtain from (3) the equation (see detailed discussion in Granovsky and Zeifman (2004), Zeifman et al. (2006))

$$\frac{d\mathbf{z}}{dt} = B(t)\mathbf{z}(t) + \mathbf{f}(t), \quad (4)$$

where $\mathbf{f}(t) = (a_{10}, a_{20}, \ldots)^T$, $\mathbf{z}(t) = (p_1, p_2, \ldots)^T$,

$$B = \begin{pmatrix} a_{11} - a_{10} & a_{12} - a_{10} & \cdots & a_{1r} - a_{10} & \cdots \\ a_{21} - a_{20} & a_{22} - a_{20} & \cdots & a_{2r} - a_{20} & \cdots \\ a_{31} - a_{30} & a_{32} - a_{30} & \cdots & a_{3r} - a_{30} & \cdots \\ \cdots & & & & \\ a_{r1} - a_{r0} & a_{r2} - a_{r0} & \cdots & a_{rr} - a_{r0} & \cdots \\ \cdots\cdots\cdots & & & & \end{pmatrix}. \quad (5)$$

Let $\bar{X} = \bar{X}(t)$ be the "perturbed" Markov chain with the state probabilities $\bar{p}_i(t)$, the transposed intensity matrix $\bar{A}(t) = \left(\bar{a}_{ij}(t)\right)_{i,j=0}^{\infty}$ and so on. Denote by $E(t,k) = E\{X(t)|X(0) = k\}$ the mean (the mathematical expectation) of the process at the moment t under the initial condition $X(0) = k$.

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