



On the probability of conjunctions of stationary Gaussian processes



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ARTICLE INFO

Article history:

Received 29 October 2013

Received in revised form 12 January 2014

Accepted 3 February 2014

Available online 14 February 2014

MSC:

primary 60G15

secondary 60G70

Keywords:

Stationary Gaussian processes

Order statistics processes

Conjunction

Extremes

Berman sojourn limit theorem

Generalized Pickands constant

ABSTRACT

Let $\{X_i(t), t \geq 0\}$, $1 \leq i \leq n$ be independent centered stationary Gaussian processes with unit variance and almost surely continuous sample paths. For given positive constants u , T , define the set of conjunctions $C_{[0,T],u} := \{t \in [0, T] : \min_{1 \leq i \leq n} X_i(t) \geq u\}$. Motivated by some applications in brain mapping and digital communication systems, we obtain exact asymptotic expansion of $\mathbb{P}\{C_{[0,T],u} \neq \emptyset\}$, as $u \rightarrow \infty$. Moreover, we establish the Berman sojourn limit theorem for the random process $\{\min_{1 \leq i \leq n} X_i(t), t \geq 0\}$ and derive the tail asymptotics of the supremum of each order statistics process.

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1. Introduction and main result

Let $X_i(\mathbf{t})$ model the value of an image i at location $\mathbf{t} \in \mathbb{R}^d$, $1 \leq i \leq n$. For a given positive threshold u and a given scan set $\mathcal{T} \subset \mathbb{R}^d$, the set of conjunctions $C_{\mathcal{T},u}$ is defined by

$$C_{\mathcal{T},u} := \left\{ \mathbf{t} \in \mathcal{T} : \min_{1 \leq i \leq n} X_i(\mathbf{t}) \geq u \right\}$$

see the seminal contribution (Worsley and Friston, 2000). As mentioned in the aforementioned paper, of interest is the calculation of the probability that the set of conjunctions $C_{\mathcal{T},u}$ is not empty, i.e.,

$$p_{\mathcal{T},u} := \mathbb{P}\{C_{\mathcal{T},u} \neq \emptyset\} = \mathbb{P}\left\{ \sup_{\mathbf{t} \in \mathcal{T}} \min_{1 \leq i \leq n} X_i(\mathbf{t}) \geq u \right\}.$$

Typically, in applications such as the analysis of functional magnetic resonance imaging (fMRI) data, X_i 's are assumed to be real-valued Gaussian random fields. Approximations of $p_{\mathcal{T},u}$ are discussed for smooth Gaussian random fields in Worsley

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<http://dx.doi.org/10.1016/j.spl.2014.02.004>

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and Friston (2000), Alodat (2011) and Cheng and Xiao (2013); results for non-Gaussian random fields can be found in Alodat et al. (2010).

In this paper, we shall consider the case $d = 1$, $\mathcal{T} := [0, T]$, with $T > 0$, and that X_i 's are independent centered stationary Gaussian processes with unit variance and correlation functions $r_i(\cdot)$, $1 \leq i \leq n$ that satisfy

$$r_i(t) = 1 - C_i |t|^{\alpha_i} + o(|t|^{\alpha_i}), \quad t \rightarrow 0, \quad r_i(t) < 1, \quad \forall t \in (0, T] \quad (1)$$

for some positive constants $\alpha_i \in (0, 2]$ and C_i , $1 \leq i \leq n$. Further, we assume that X_i 's have almost surely continuous sample paths. Since the calculation of $p_{\mathcal{T}, u}$ is not possible in general, we shall investigate below the exact asymptotic behavior of $p_{\mathcal{T}, u}$ as $u \rightarrow \infty$. Although $\{\min_{1 \leq i \leq n} X_i(t), t \geq 0\}$ is not a Gaussian process when $n \geq 2$, as shown in Worsley and Friston (2000), it happens that it is possible to adapt techniques used in the theory of Gaussian processes and random fields to this class of processes. Motivated by a recent paper of Albin and Choi (2010) and the extremal theory for stationary Gaussian processes developed mainly by Berman and Albin (see Berman, 1982, 1992, Albin, 1990 and Albin and Jarušková, 2003), we shall derive an asymptotic expansion for $p_{\mathcal{T}, u}$ as $u \rightarrow \infty$, by following the ideas of Albin and Choi (2010).

For the formulation of our main result we need to introduce some notation. Let $\{B_{\alpha_i}(t), t \geq 0\}$, $1 \leq i \leq n$ be mutually independent standard fractional Brownian motions with Hurst indexes $\alpha_i/2 \in (0, 1]$, $1 \leq i \leq n$, respectively, i.e., B_{α_i} is a centered Gaussian process with continuous sample paths and covariance function

$$\text{Cov}(B_{\alpha_i}(t), B_{\alpha_i}(s)) = \frac{1}{2} (t^{\alpha_i} + s^{\alpha_i} - |t - s|^{\alpha_i}), \quad t, s > 0, \quad 1 \leq i \leq n.$$

Next define

$$\mathcal{Z}(t) := \min_{1 \leq i \leq n} \left(\left(\sqrt{2} B_{\alpha_i}(C_i^{1/\alpha_i} t) - C_i t^{\alpha_i} \right) \mathbf{1}(\alpha_i = \alpha_{\min}) + E_i \right), \quad t \geq 0, \quad \alpha_{\min} := \min_{1 \leq i \leq n} \alpha_i, \quad (2)$$

where $\mathbf{1}(\cdot)$ denotes the indicator function, and E_i 's are mutually independent unit exponential random variables being further independent of B_{α_i} 's. Finally, let $\mathcal{H}_{\alpha_1, \dots, \alpha_n}(C_1, \dots, C_n) \in (0, \infty)$ denote the *generalized Pickands constant*, determined by

$$\mathcal{H}_{\alpha_1, \dots, \alpha_n}(C_1, \dots, C_n) = \lim_{a \downarrow 0} \frac{1}{a} \mathbb{P} \left\{ \max_{k \geq 1} \mathcal{Z}(ak) \leq 0 \right\}. \quad (3)$$

The following theorem constitutes our principle result.

Theorem 1.1. Let $\{X_i(t), t \geq 0\}$, $1 \leq i \leq n$ be mutually independent centered stationary Gaussian processes with unit variance and correlation functions satisfying (1). Then, for any $T > 0$

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} \min_{1 \leq i \leq n} X_i(t) > u \right\} = \mathcal{H}_{\alpha_1, \dots, \alpha_n}(C_1, \dots, C_n) T u^{\frac{2}{\alpha_{\min}}} \frac{\exp(-nu^2/2)}{(2\pi)^{n/2} u^n} (1 + o(1)), \quad u \rightarrow \infty, \quad (4)$$

where $\mathcal{H}_{\alpha_1, \dots, \alpha_n}(C_1, \dots, C_n) \in (0, \infty)$ is defined in (3).

The organization of the paper. Section 2 presents brief discussions and shows the validity of the Berman sojourn limit theorem for the random process $\{\min_{1 \leq i \leq n} X_i(t), t \geq 0\}$. Additionally, utilizing the fact that minimum is a particular case of the order statistics, in Theorem 2.2 we get a counterpart of Theorem 1.1 for order statistics processes. The case of non-standard stationary Gaussian processes is treated in Theorem 2.3. Section 3 contains all the proofs.

2. Discussions and extensions

In his seminal contribution (Pickands, 1969) J. Pickands III established the exact asymptotic tail behavior of the supremum of the stationary Gaussian process $\{X_1(t), t \in [0, T]\}$ under the condition (1), using a double-sum method. The first crucial step to that result is the celebrated Pickands lemma which states that, for any positive constant S

$$\mathbb{P} \left\{ \sup_{t \in [0, u^{-\frac{2}{\alpha_1}} S]} X_1(t) > u \right\} = \mathcal{H}_{\alpha_1} \left[0, C_1^{\frac{1}{\alpha_1}} S \right] \Psi(u) (1 + o(1)), \quad u \rightarrow \infty,$$

where $\Psi(\cdot)$ is the survival function of an $N(0, 1)$ random variable and

$$\mathcal{H}_{\alpha_1}[0, S] = \mathbb{E} \left\{ \exp \left(\sup_{t \in [0, S]} \left(\sqrt{2} B_{\alpha_1}(t) - t^{\alpha_1} \right) \right) \right\} \in (0, \infty).$$

Recall that $\Psi(u) = \exp(-u^2/2)/\sqrt{2\pi}u^2(1 + o(1))$ as $u \rightarrow \infty$.

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