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A probabilistic approach for gradient estimates on time-inhomogeneous manifolds

Li-Juan Cheng

School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China

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ABSTRACT

Gradient estimates of Hamilton and Li–Yau types for positive solutions to the heat equation are established from a probabilistic viewpoint, which simplifies the proofs of some results of Sun (2011).

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1. Introduction

In this paper, we establish gradient estimates on time-inhomogeneous manifolds by stochastic analysis methods. It is well known that gradient estimates, known as differential Harnack inequalities, are powerful tools both in geometric and stochastic analyses. For example, Hamilton (1995) established differential Harnack inequalities for the scalar curvature under Ricci flow, which is applied to singularity analysis; Perelman (2002) used a differential Harnack inequality in his proof of the Poincaré conjecture.

There have been many works on gradient estimates along the Ricci flow or the conjugate Ricci flow for all positive solutions to the heat equation; see Bailesteanu et al. (2010), Cao (2008), Cao and Hamilton (2009), Kuang and Zhang (2008), Liu (2009) and Zhang (2006) for instance. Under some curvature constraints, Sun (2011) generalized the work on Li–Yau type gradient estimates to general geometric flows by using the method of Li and Yau (1986). We will revisit these problems under general geometric flows from a probabilistic viewpoint. When the metric is independent of t, this point of view has been worked out well for local estimates of positive harmonic functions (Arnaudon et al., 2007) and for Li–Yau type gradient estimates (Arnaudon and Thalmaier, 2010) on Riemannian manifolds.

Let M be a d-dimensional differentiable manifold without boundary equipped with a $C^{1,\infty}$-family of complete Riemannian metrics $(g_t)_{t\in[0,T]}$, $T \in (0, T_c)$, where T_c is the time when the curvature possibly blows up. In light of the work of Perelman, it is convenient to consider the heat equation and the evolution of the metric backwards in time. Indeed, by a

E-mail addresses: chenglj@mail.bnu.edu.cn, xiaochenglijuan@163.com.

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time reversal, it leads to our familiar forward form. Suppose a smooth positive function $u : M \times [0, T] \to \mathbb{R}$ satisfies the (backward) heat equation

$$\partial_t u(x, \cdot)(t) = -\frac{1}{2}\Delta_t u(\cdot, t)(x) \tag{1.1}$$

on $M \times [0, T]$, where Δ_t is the Laplacian associated with the metric g_t. This paper is devoted to Hamilton and Li–Yau type gradient estimates for u by constructing suitable semimartingales. This idea is inspired by Arnaudon and Thalmaier (2010). To explain it, we investigate the Li–Yau gradient estimate on a compact manifold carrying a (backward) Ricci flow for example. Here and in what follows, the Ricci flow means (probabilistic convention):

$$\partial_t g(x, \cdot)(t) = \mathrm{Ric}(x, t), \quad (x, t) \in M \times [0, T], \tag{1.2}$$

where $\mathrm{Ric}(\cdot, t) = \mathrm{Ric}_t$ is the Ricci curvature tensor with respect to the metric g_t. Moreover, suppose that there exists constant $k > 0$ such that

$$0 \le \mathrm{Ric}_t \le k$$

holds on $M \times [0, T]$. Let X_t be a g_t-Brownian motion (see Arnaudon et al., 2008, for the construction), which is ensured to be non-explosive under Ricci flow (see Kuwada and Philipowski, 2011). Let

$$\hat{S}_t = h_t\left(\frac{|\nabla^t u|_t^2}{u}(X_t, t) - \Delta_t u(\cdot, t)(X_t)\right) + n\dot{h}_t u(X_t, t), \tag{1.3}$$

where ∇^t, $|\cdot|_t$ are respectively the gradient operator and the norm associated with the metric g_t and h_t is the solution to the following equation

$$\dot{h}_t = -h_t[c_1(T-t)^{-1} + c_2 k], \quad h_0 = 1$$

for some positive constants c_1, c_2. Then,

$$\hat{S}_t = h_t\left\{\frac{|\nabla^t u|_t^2}{u}(X_t, t) - \Delta_t u(\cdot, t)(X_t) - nu(X_t, t)[c_1(T-t)^{-1} + c_2 k]\right\}. \tag{1.4}$$

So, if we can choose suitable constants c_1 and c_2 in (1.4) such that $\hat{S}_t$ is a submartingale, then the Li–Yau gradient estimate can be derived directly; see the proof of Lemma 3.2 below. Note that when reduced to the constant metric case, the classical Li–Yau inequality says that $\frac{|\nabla u|^2}{u^2} - \frac{\Delta u}{u}$ can be dominated by $\frac{n}{t}$ and it does not need the curvature condition "$\mathrm{Ric} \le k$". However, in our setting, we need it to control additional terms from the time derivative of the metric, e.g. the derivative of $\Delta_t u$ with respect to t. This is the main difficulty for us to overcome. Compared with former results, the coefficients appearing in the gradient inequalities under general geometric flows are more explicit.

The rest of the paper is organized as follows. In Section 2, we generalize Bailesteanu–Cao–Pulemotov's work (Bailesteanu et al., 2010) on Hamilton type gradient estimates under Ricci flow to general geometric flows. Then, in Section 3, we investigate Li–Yau type gradient inequalities. In Section 4, we give an application to Ricci flow.

We end this section by making some conventions on the notation. Throughout the whole paper, the norm $|\cdot|_t$ and the metric $\langle\cdot,\cdot\rangle_t$ are also used for that on vector bundles associated with g_t. For any two-tensor T_t and any function f, we write $\mathrm{T}_t \ge f$, if $\mathrm{T}_t(X, X) \ge f\,\langle X, X\rangle_t$.

2. Gradient estimates of Hamilton type

For simplicity, we introduce some notations first. For any subset $D \subset M \times [0, T]$ and function f defined on $M \times [0, T]$, let $\|f\|_D = \sup_{(x,t)\in D} |f|$, and for $X, Y \in TM$, let

$$\mathcal{R}_t(X, Y) = \mathrm{Ric}_t(X, Y) - \partial_t g_t(X, Y).$$

The Hamilton type inequality, i.e. an estimate in terms of the uniform norm of u, on compact manifolds is presented as follows.

Theorem 2.1 (*Gradient Inequality of Hamilton Type*)**.** *Let M be a compact manifold such that $\mathcal{R}_t \ge -k(t)$ for some $k \in C([0, T])$. Suppose that u is a positive solution to the heat equation (1.1) on $M \times [0, T]$. Then, for $(x, t) \in M \times [0, T)$, we have*

$$\frac{|\nabla^t u|_t^2}{u^2}(x, t) \le \frac{2}{\int_t^T e^{-\int_t^s k(r)\mathrm{d}r}\mathrm{d}s} \log \frac{\|u\|_{M\times[0,T]}}{u(x, t)}.$$

The following two lemmas are essential to the proof of Theorem 2.1. First, we introduce some basic formulae for solutions to the heat equation.

Lemma 2.2. *Let $u = u(x, t)$ be a positive solution to the heat equation (1.1) on $M \times [0, T]$. Then the following equations hold:*

$$\left(\frac{1}{2}\Delta_t + \partial_t\right)(u \log u) = \frac{1}{2}\frac{|\nabla^t u|_t^2}{u},$$

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