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Cumulative distribution functions and moments of lattice polynomials

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Abstract

We give the cumulative distribution functions, the expected values, and the moments of lattice polynomials when regarded as real functions. Since lattice polynomial functions include order statistics, our results encompass the corresponding formulas for order statistics.

Keywords: Lattice polynomial; Order statistic; Cumulative distribution function; Moment; Reliability

1. Introduction

Order statistics have been deeply investigated from a probabilistic viewpoint. Their cumulative distribution functions (c.d.f.'s) as well as their expected values and moments are now well known. For theoretical developments, see e.g. the remarkable monograph by David and Nagaraja (2003).

Order statistics can be naturally extended to nonsymmetric real functions called lattice polynomial functions (Birkhoff, 1967). Roughly speaking, an n-ary *lattice polynomial* function is defined from any well-formed expression involving n real variables $x_1, \ldots, x_n$ linked by the lattice operations $\wedge = \min$ and $\vee = \max$ in an arbitrary combination of parentheses. For instance,

$$p(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee x_3$$

is a 3-ary lattice polynomial function.

In this note we give closed-form formulas for the c.d.f. of any lattice polynomial function in terms of the c.d.f.'s of its input variables. More precisely, considering a lattice polynomial function $p : \mathbb{R}^n \to \mathbb{R}$ and n independent random variables $X_1, \ldots, X_n$, X_i $(i = 1, \ldots, n)$ having c.d.f. $F_i(x)$, we give formulas for the c.d.f. of $Y_p := p(X_1, \ldots, X_n)$. We also yield formulas for the expected value $\mathbf{E}[g(Y_p)]$, where g is any measurable function. The special cases $g(x) = x$, x^r, $[x - \mathbf{E}(Y_p)]^r$, and e^{tx} give, respectively, the expected value, the raw moments, the central moments, and the moment-generating function of Y_p.

As the order statistics are exactly the symmetric lattice polynomial functions (Marichal, 2002), we retrieve, as a corollary, the well-known formulas of the c.d.f.'s and the moments of the order statistics.

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As we will see in Section 2, any n-ary lattice polynomial function is completely determined by its values at the 2^n elements of $\{0,1\}^n$. In turn, the formulas we provide are expressed in terms of those values.

This note is organized as follows. In Section 2 we recall the basic material related to lattice polynomial functions and order statistics. In Section 3 we provide the announced results. In Section 4 we investigate the particular case where the input random variables are identically distributed. Finally, in Section 5 we provide an application of our results to the reliability analysis of coherent systems.

Lattice polynomial functions play an important role in conjoint measurement theory. Ovchinnikov (1998, Theorem 5.3) proved that the n-ary lattice polynomial functions are exactly those functions $f:\mathbb{R}^n\to\mathbb{R}$ which are continuous and *order-invariant* in the sense that, for any increasing bijection $\phi:\mathbb{R}\to\mathbb{R}$, we have

$$f[\phi(x_1),\dots,\phi(x_n)]=\phi[f(x_1,\dots,x_n)].$$

Marichal (2002, Corollary 4.4) showed that, in this axiomatization, continuity can be replaced with nondecreasing monotonicity in each variable.

2. Lattice polynomial functions

As we shall be concerned with lattice polynomials of random variables, we will not consider lattice polynomials on a general lattice, but simply on the real line $\mathbb{R}$, which is a particular lattice. The lattice operations $\wedge$ and $\vee$ will then represent the minimum and maximum operations, respectively. To simplify the notation, we will also set $[n]:=\{1,\dots,n\}$ for any integer $n\geqslant 1$.

Let us first recall the definition of a lattice polynomial (with real variables); see e.g. Birkhoff (1967, Section II.5).

Definition 1. Given a finite collection of variables $x_1,\dots,x_n\in\mathbb{R}$, a *lattice polynomial* in the variables $x_1,\dots,x_n$ is defined as follows:

(1) the variables $x_1,\dots,x_n$ are lattice polynomials in $x_1,\dots,x_n$;
(2) if p and q are lattice polynomials in $x_1,\dots,x_n$, then $p\wedge q$ and $p\vee q$ are lattice polynomials in $x_1,\dots,x_n$;
(3) every lattice polynomial is formed by finitely many applications of the rules (1) and (2).

When two different lattice polynomials p and q in the variables $x_1,\dots,x_n$ represent the same function from $\mathbb{R}^n$ to $\mathbb{R}$, we say that p and q are equivalent and we write $p=q$. For instance, $x_1\vee(x_1\wedge x_2)$ and x_1 are equivalent.

Because $\mathbb{R}$ is a distributive lattice, any lattice polynomial function can be written in *disjunctive* and *conjunctive* forms as follows; see e.g. Birkhoff (1967, Section II.5).

Proposition 2. *Let $p:\mathbb{R}^n\to\mathbb{R}$ be any lattice polynomial function. Then there are nonconstant set functions $v:2^{[n]}\to\{0,1\}$ and $w:2^{[n]}\to\{0,1\}$, with $v(\emptyset)=0$ and $w(\emptyset)=1$, such that*

$$p(x)=\bigvee_{\substack{S\subseteq[n]\\ v(S)=1}}\bigwedge_{i\in S}x_i=\bigwedge_{\substack{S\subseteq[n]\\ w(S)=0}}\bigvee_{i\in S}x_i.$$

The set functions v and w that disjunctively and conjunctively generate the polynomial function p in Proposition 2 are not unique. For example, as we have already observed above, we have

$$x_1\vee(x_1\wedge x_2)=x_1=x_1\wedge(x_1\vee x_2).$$

However, it can be shown (Marichal, 2002) that, from among all the possible set functions that disjunctively generate a given lattice polynomial function, only one is nondecreasing. Similarly, from among all the possible set functions that conjunctively generate a given lattice polynomial function, only one is nonincreasing. These particular set functions are given by

$$v(S)=p(\mathbf{1}_S)\quad\text{and}\quad w(S)=p(\mathbf{1}_{[n]\setminus S}),$$