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When does $E(X^k \cdot Y^l) = E(X^k) \cdot E(Y^l)$ imply independence?

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Abstract

Our aim is to find conditions on random variables X and Y that either ensure or do not ensure that the validitiy of the equation in the title (for all positive integers k and l) implies that X and Y are independent.

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1. Introduction

Throughout this paper we consider real-valued random variables defined on the same probability space. It is well known that two random variables X and Y are independent, if and only if

$$f_{(X,Y)}(s,t) = f_X(s) \cdot f_Y(t), \quad s,t \in \mathbf{R},$$

where f_X and f_Y are the characteristic functions of X and Y, respectively, on the real line $\mathbf{R}$ while $f_{(X,Y)}$ is the characteristic function of the random vector (X, Y).

The analogous statement involving moments instead of characteristic functions is not true in general. There exist random variables X and Y having moments of all orders such that the equation

$$E(X^k \cdot Y^l) = E(X^k) \cdot E(Y^l) \tag{1}$$

holds for all k and l in $\mathbf{N}$ (the positive integers), yet X and Y are not independent.

The aim of the present paper is to find conditions on X and Y that either ensure or do not ensure that the validitiy of (1) implies independence. In some of our results we pose no restriction on the ranges of the random variables while in others we assume that their ranges lie in certain subsets of $\mathbf{R}$ (e.g., nonnegative or integer-valued random variables). This part was motivated by the following question of L. Heinrich:

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Is it true that the validitiy of (1) for all k and l implies independence when we additionally know that X and Y take only nonnegative integer values?

As we shall see, the answer to this question is "no" (by Theorem 5).

Because of our Theorems 3 and 4, it is of interest to be able to decide whether a given moment sequence is determinate or indeterminate. In the indeterminate case, it is also of interest to exhibit explicitly a whole line segment of measures having the same moment sequence. For these topics, see Stoyanov (2004) and references therein.

In the sequel, we assume that the random variables in question have moments of all orders. Let $\mathbf{N}_0$ be the set $\{0\} \cup \mathbf{N}$. For every topological space K, let $\mathscr{B}(K)$ be the set of all Borel subsets of K. Let $\delta(\cdot,\cdot)$ be the Kronecker delta, and for every set A let 1_A be its indicator function.

2. Results

Though the first two theorems are not new, we present them with proofs for the sake of completeness. (Rényi, 1962, asserts that Theorem 1 was proved by Kantorovich, 1929, and Theorem 2 by Kac (no reference given) for bounded random variables.)

Theorem 1. *Let X and Y be discrete random variables with values in finite sets with m and n elements, respectively. If the Eq.* (1) *holds for all k in $\{1,\ldots,m-1\}$ and l in $\{1,\ldots,n-1\}$ then X and Y are independent.*

Proof. Let S be an arbitrary subset of $\mathbf{R}$ with N elements. The restrictions of the monomials $1, t, \ldots, t^{N-1}$ form a basis of the real linear space of real-valued functions on S. Consequently, (the restriction to S of) an arbitrary polynomial is a linear combination of these (restrictions of) monomials. Using this, it is easy to see that the Eq. (1) holds for all k and l in $\mathbf{N}$. Therefore, the theorem is an immediate consequence of Theorem 2. □

Though the next theorem follows from Theorem 3, we present a short proof using characteristic functions. Recall that the characteristic function f of a random variable X is called *analytic* if there is a positive real ρ such that the function $f_{|]-\rho,\rho[}$ extends to a holomorphic function in the disk $\{z \in \mathbf{C} : |z| < \rho\}$. If f is analytic then (for such ρ)

$$f(t) = \sum_{k=0}^{\infty} E(X^k) \cdot \frac{(it)^k}{k!}, \quad |t| < \rho$$

and moreover, the distribution of f is uniquely determined by the moments of X (see Bisgaard and Sasvári, 2000, Section 1.12 and Billingsley, 1995, Section 30). The converse is not true. For example, define a sequence $t = \{t_n\}_0^\infty$ by $t_n = e^{n^2/2}$ ($n \in \mathbf{N}_0$). Then t is an indeterminate moment sequence (Berg et al., 1984, 6.4.6) and has (see Akhiezer, 1965) a representing measure σ with $\sigma(\{0\}) > 0$ and the polynomials dense in $\mathscr{L}^2(\sigma)$. Let $s = \{s_n\}_0^\infty$ be the moment sequence of the measure $(1 - \sigma(\{0\}))^{-1} 1_{\mathbf{R}\setminus\{0\}}(x)\,d\sigma(x)$. Then s is determinate (see Berg and Christensen, 1981) but the corresponding characteristic function is not analytic.

(Another possibility is to choose a sequence $c = \{c_n\}_1^\infty$ of positive reals and then to let s_n be the upper left entry in the nth power of the infinite matrix

$$\begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & c_1 & & & \\ & c_1 & 0 & 1 & & \\ & & 1 & 0 & c_2 & \\ & & & c_2 & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix},$$

where entries neither shown nor indicated by dots are 0. By a result of Carleman (see Akhiezer, 1965), the sequence s is then determinate, while it is clear that s is not analytic if the sequence c grows fast enough (for example, $c_n = n^\lambda$ with $\lambda > 2$).)