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Statistics & Probability Letters 76 (2006) 1164–1169

STATISTICS & PROBABILITY LETTERS

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An asymptotic estimate for Brownian motion with drift

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Received 21 September 2004; received in revised form 22 November 2005
Available online 5 January 2006

Abstract

We apply an Abelian theorem, due to Berg, to determine the asymptotic behaviour of $\mathbb{P}[\xi_t > x]$ as $x^2t^{-1} - \gamma' x \uparrow \infty$ when ξ is the range of Brownian motion with positive drift $\gamma < \gamma'$. The method is simple, general, and yields a sharp error bound.

MSC: primary 60J65; secondary 41A60; 34B24

Keywords: Tauberian theorem; Brownian motion; Hitting times

Notation: All processes start at zero: B is Brownian motion, $X^\bullet$ (resp. X°) denotes the maximum (resp. minimum) of X, while $C, C' \ldots$ represent generic strictly positive constants. We always take $\tau = t/x^2 \downarrow 0$.

In Daudin et al. (2003) the authors obtain a large deviation estimate for the range of Brownian motion with positive drift. This is the process $\xi = \xi^{(\gamma)} = (X - X^\circ)^\bullet$, where $X_t = B_t + \gamma t$, and they find

$$\mathbb{P}[\xi_1 > x] \sim \frac{2}{x}\sqrt{\frac{2}{\pi}}\mathrm{e}^{-(x-\gamma)^2/2}, \quad x \uparrow \infty. \tag{1}$$

Their paper offers two *ad hoc* proofs: one using special functions, another via path decomposition of ξ. Here we point out that (1) follows from an Abelian theorem formulated by Berg (1974) Section 49 pp. 112–113. This was applied by Csáki (1989) to a problem with Brownian scaling but our result illustrates its wider utility. The statement is as follows.

Theorem. *$\exists G$ holomorphic on $\Re(z) > \gamma x > 0$ such that for any constant $\gamma' > \gamma$*

$$\begin{aligned}\mathbb{P}[\xi_t > x] &= \frac{1}{\sqrt{2\tau\pi}}\mathrm{e}^{-(1-\tau\gamma x)^2/2\tau}G(1/2\tau)(1 + \mathrm{O}(\tau))\\ &= 2\sqrt{\frac{2\tau}{\pi}}\frac{\mathrm{e}^{-(1-\tau\gamma x)^2/2\tau}}{(1-\tau\gamma x)(1+\tau\gamma x)^2}(1+\mathrm{O}(\tau)), \quad \tau^{-1} - \gamma' x \uparrow \infty,\end{aligned} \tag{2}$$

with error constant independent of x.

doi:10.1016/j.spl.2005.12.023

The proof is deferred. We remark that this is a Tauberian theorem since G, given explicitly at (10) below, is related to the Laplace transform computed by Williams (1976):

$$\int_0^\infty e^{-zt}\mathbb{P}[\xi_t > x]\,dt = \frac{\mu e^{\gamma x}}{z[\mu\cosh \mu x + \gamma \sinh \mu x]}, \quad \mu = \sqrt{2z+\gamma^2}. \tag{3}$$

Applying Berg's theorem directly to (3) gives (2) for x fixed and $t \downarrow 0$, but to obtain our more general statement, and its immediate corollary (1), we must control the error as x varies. For this, we rework the proof of Berg's result starting from the following elementary observation (cf. Berg (1974) 49.3).

Lemma. *Let G be holomorphic on $\Re z > \rho_0 > 0$, and suppose there exist $\alpha \geqslant 0$, $\beta > 1$, and $\eta > \frac{1}{2}$ such that for all $\rho > \rho_0$ we have*:

$$|G(\rho + iu)e^{-\alpha|u|/\sqrt{\rho}}| \leqslant C_{A,\alpha}|G(\rho)| \quad \text{uniformly in } |u| \geqslant \rho^\eta \tag{A_α}$$

$$|G''(\rho + iu)| \leqslant C_{B,\beta}|\rho^{-\beta}G(\rho)| \quad \text{uniformly in } |u| \leqslant \rho^\eta. \tag{B_β}$$

Then

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty G(\rho + i\sqrt{\rho}u)e^{-u^2}\,du = G(\rho)(1 + O(\rho^{1-\beta})), \quad \rho \uparrow \infty. \tag{4}$$

Proof. We prove (4) in the form

$$\frac{1}{\sqrt{\pi\rho}}\int_{-\infty}^\infty [G(\rho + iu) - G(\rho)]e^{-u^2/\rho}\,du = O(\rho^{1-\beta}G(\rho)), \quad \rho \uparrow \infty,$$

in two steps. First, we can ignore the contribution from $\mathbb{R}\backslash(-\rho^\eta, \rho^\eta)$ since (e.g.)

$$\left|\frac{1}{\sqrt{\rho}}\int_{\rho^\eta}^\infty G(\rho + iu)e^{-u^2/\rho}\,du\right| \leqslant C_{A,\alpha}|G(\rho)|\int_{\rho^\eta}^\infty e^{\alpha u/\sqrt{\rho}}e^{-u^2/\rho}\frac{du}{\sqrt{\rho}}$$
$$= C_{A,\alpha}|G(\rho)|e^{\alpha^2/4}\int_{\rho^{\eta-1/2}}^\infty e^{-(u-1/2\alpha)^2}\,du = o(|G(\rho)|e^{-(1/2)\rho^{2\eta-1}}).$$

Next, using Taylor expansion and (B_β), we obtain

$$\frac{1}{\sqrt{\pi\rho}}\int_{-\rho^\eta}^{\rho^\eta}|G(\rho + iu) - G(\rho)|e^{-u^2/\rho}\,du \leqslant \frac{\rho}{\sqrt{\pi}}\sup_{|u|\leqslant\rho^\eta}|G''(\rho + iu)|\int_0^{\rho^{\eta-1/2}}u^2e^{-u^2}\,du$$
$$\leqslant \frac{C_{B,\beta}}{\sqrt{\pi}}|G(\rho)|\rho^{1-\beta}\int_0^\infty u^2e^{-u^2}\,du = \frac{C_{B,\beta}}{4}|G(\rho)|\rho^{1-\beta}. \qquad \square$$

Remarks 5. (1) For $\rho \geqslant \rho_0(\alpha, \eta, C_{A,\alpha})$ the error constant in (4) is bounded by $\frac{1}{2}C_{B,\beta}$.

(2) If G is real on $[\rho_0, \infty)$ then $E = G'/G = o(1)$. For if $\inf_{\rho\geqslant\rho_0}E_\rho^2 \geqslant \delta > 0$ then, from $|E' + E^2| = |G''/G| \leqslant C_{B,\beta}\rho^{-\beta}$, we deduce $E' \leqslant -\frac{1}{2}E^2$ eventually and hence $E \downarrow 0$. This contradiction shows $E_{\rho_n}^2 \downarrow_n 0$ for some $\rho_n \uparrow \infty$ and, applying the above bound whenever $E' > 0$, we obtain $\sup_{\rho\geqslant\rho_n}E_\rho^2 \leqslant \max(E_{\rho_n}^2, C_{B,\beta}\rho_n^{-\beta}) = o(1)$.

(3) Given $\varepsilon > 0$, we deduce existence of $\rho_\varepsilon \geqslant \rho_0$ such that $|G(\rho)| \leqslant |G(\rho_\varepsilon)|e^{\varepsilon\rho}$ if $\rho \geqslant \rho_\varepsilon$.

(4) One may assume $\frac{1}{2} < \eta < 1$ provided G is real on $[\rho_0, \infty)$ and $\alpha > 0$. In fact, by Taylor expansion, remark (2), and condition (B_β), we have

$$G(\rho + iu) = G(\rho) + iuG'(\rho) - \tfrac{1}{2}u^2G''(\rho + i\theta_u u) = O(\rho^{2\eta}G(\rho))$$

uniformly in $u < \rho^\eta$; so for $\frac{1}{2} < \eta' < 1 \leqslant \eta$ and any $\varepsilon > 0$

$$|G(\rho + iu)| \leqslant C\rho^{2\eta}G(\rho) \leqslant Ce^{\varepsilon\rho^{\eta'-1/2}}G(\rho) \leqslant Ce^{\varepsilon|u|/\sqrt{\rho}}G(\rho)$$

uniformly in $\rho^{\eta'} \leqslant |u| < \rho^\eta$ as $\rho \uparrow \infty$. Meaning (A_α) holds whenever $|u| \geqslant \rho^{\eta'}$, though perhaps with a larger constant.