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Almost sure max-limits for nonstationary Gaussian sequence

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Abstract

We obtain some almost sure limit theorems for a standardized nonstationary Gaussian sequence under some mild conditions.

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1. Introduction and results

In 1988, a new chapter of limit theorem has been discovered, which is called almost sure central limit theorem (ASCLT). Many profound results have been obtained for independent and dependent random variables since 1988. Fahrner and Stadtmüller (1998), Cheng et al. (1998), Fahrner (2000) and Stadtmüller (2002) investigated almost sure limit theorem for the maxima of i.i.d. random variables. Csáki and Gonchigdanzan (2002) got an almost sure limit theorem for the maximum of stationary Gaussian sequence.

In this paper, we study almost sure limit theorems of nonstationary Gaussian sequence under some mild conditions. In the sequel, A denotes a positive constant whose value may vary from line to line.

Theorem 1.1. *Let $\{\xi_n\}$ be a sequence of nonstationary Gaussian random variables with zero mean, unit variance and covariance matrix (r_{ij}) such that $|r_{ij}| \leqslant \rho_{|i-j|}$ for $i \neq j$ where $\rho_n < 1$ for all $n \geqslant 1$ and*

$$\rho_n \log n \leqslant \frac{A}{(\log \log n)^{1+\varepsilon}}. \tag{1.1}$$

Let the constants $\{u_{ni}\}$ be such that $\sum_{i=1}^{n}(1-\Phi(u_{ni}))$ is bounded and $\lambda_n = \min_{1 \leqslant i \leqslant n} u_{ni} \geqslant c(\log n)^{1/2}$ for some $c>0$. If $\sum_{i=1}^{n}(1-\Phi(u_{ni})) \to \tau$ as $n \to \infty$ for some $\tau \geqslant 0$, then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I_{\{\cap_{i=1}^{k}(\xi_i \leqslant u_{ki})\}} = \exp(-\tau) \quad a.s.,$$

where I is indicator function.

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Let $\eta_i = \xi_i + m_i$ be a normal sequence, where $\{\xi_n\}$ is a standardized Gaussian sequence with covariance r_{ij}. The constants m_i satisfy

$$\beta_n = \max_{1 \leqslant i \leqslant n} |m_i| = \mathrm{o}(\log n)^{1/2}) \quad \text{as} \quad n \to \infty \tag{1.2}$$

m_n^* is defined by

$$\frac{1}{n}\sum_{i=1}^{n} \exp(a_n^*(m_i - m_n^*) - 1/2(m_i - m_n^*)^2) \to 1 \quad \text{as} \quad n \to \infty \tag{1.3}$$

in which $a_n^* = a_n - \log\log n/2a_n$, $a_n = (\log n)^{1/2}$, $b_n = a_n - (\log\log n + \log 4\pi)/2a_n$.

The following theorem is obtained from Theorem 1.1 and Theorem 6.2.1 in Leadbetter et al. (1983).

Theorem 1.2. *Let $\{\eta_n\}$ be defined as above by $\eta_i = \xi_i + m_i$ where $\{\xi_n\}$ is a standardized Gaussian sequence with covariances r_{ij} such that $|r_{ij}| \leqslant \rho_{|i-j|}$ for $i \neq j$ with $\rho_n < 1$ and condition* (1.1) *is satisfied. Suppose that $\{m_i\}$ satisfy* (1.2) *and m_n^* be defined by* (1.3). *Then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I_{\{a_k(\max_{1\leqslant i\leqslant k} \eta_i - b_k - m_k^*) \leqslant x\}} = \exp(-\mathrm{e}^{-x}) \quad a.s.$$

Xie (1984) got the asymptotic distribution of extremes in nonstationary Gaussian sequence:

Theorem A. *Suppose that $\{\xi_n\}$ is a Gaussian sequence, $E\xi_n = 0$, $D\xi_n = 1$, $r_{ij} = E\xi_i\xi_j$, and if*

(1) $\delta = \sup_{i<j} |r_{ij}| < 1$.

(2) *For some $\gamma > 2(1+\delta)/(1-\delta)$ and some positive number d, we have*

$$\frac{1}{n^2} \sum_{1 \leqslant i < j \leqslant nd} |r_{ij}| \log(j-i) \exp(\gamma |r_{ij}| \log(j-i)) \to 0 \quad as\ n \to \infty$$

Put $a_n = (2\log n)^{-1/2}$, $b_n = (2\log n)^{1/2} - (\log\log n + \log 4\pi)/(2(2\log n)^{1/2})$. Then for the positive integer sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n/n = d$, we have

$$P(M_{t_n} \leqslant a_n x + b_n) \to \exp(-d\mathrm{e}^{-x}) \quad as\ n \to \infty.$$

We extend this result to the case of almost sure limit.

Theorem 1.3. *Let $\{\xi_n,\ n \geqslant 1\}$ be a nonstationary standardized Gaussian sequence with covariance $r_{ij} = \mathrm{cov}(\xi_i, \xi_j)$ and satisfy the following conditions*:

(1) $\delta = \sup_{i<j} |r_{ij}| < 1$.

(2) *For some $\gamma > 2(1+\delta)/(1-\delta)$ and some positive number d, we have*

$$\frac{1}{n^2} \sum_{1 \leqslant i < j \leqslant nd} |r_{ij}| \log(j-i) \exp(\gamma |r_{ij}| \log(j-i)) \leqslant \frac{A}{(\log\log n)^{1+\varepsilon}}$$

for some positive numbers A and ε.

Let a_n and b_n be defined in Theorem A, $u_n = a_n x + b_n$, x is a real number. Then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I_{\{M_{t_k} \leqslant a_k x + b_k\}} = \exp(-d\mathrm{e}^{-x}) \quad a.s.,$$

where I is the indicator function.