Statistics and Probability Letters 82 (2012) 411–418

Contents lists available at SciVerse ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Weak norm inequalities for martingales and geometry of Banach spaces

Adam Osękowski

Department of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland

ARTICLE INFO

Article history:
Received 9 November 2011
Received in revised form 12 December 2011
Accepted 12 December 2011
Available online 19 December 2011

MSC:
60G42
46C15

Keywords:
Martingale
Martingale transform
Differential subordination
UMD space

ABSTRACT

Let f, g be two Hilbert-space-valued martingales such that g is differentially subordinate to f. The paper contains the proof of the estimate

$$\|g\|_{p,\infty} \leq \frac{2p(p+1)}{p-1}\|f\|_{p,\infty}, \quad 1 < p < \infty.$$

The constant is shown to be of optimal order for $p \to \infty$ and for $p \to 1$. Related results for transforms of UMD-valued martingales are also established.

1. Introduction

Martingale theory is a powerful tool in the study of the geometry of Banach spaces: see the survey of Burkholder (2001) for the wide overview of the subject. The objective of this paper is to establish some novel martingale inequalities and to explore their connections with the structure of the Banach space in which the martingales take values. In particular, this will yield some new and interesting characterizations of UMD spaces.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, filtered by $(\mathcal{F}_n)_{n\geq 0}$, a non-decreasing sequence of sub-σ-algebras of $\mathcal{F}$. Let $f = (f_n)_{n\geq 0}$, $g = (g_n)_{n\geq 0}$ be two adapted martingales with values in a given separable Banach space $(\mathbb{B}, |\cdot|)$. We define $df = (df_n)_{n\geq 0}$ and $dg = (dg_n)_{n\geq 0}$, the difference sequences of f and g, by $df_0 = f_0$, $df_n = f_n - f_{n-1}$, $n = 1, 2, \ldots$, and similarly for dg. We will use the notation $f_n^* = \sup_{0\leq k\leq n} |f_k|$, $n = 0, 1, 2, \ldots$, and for $1 \leq p < \infty$, we will write

$$\|f\|_p = \sup_{n\geq 0} \|f_n\|_p \quad \text{and} \quad \|f\|_{p,\infty} = \sup_{n\geq 0} \|f_n\|_{p,\infty} = \sup_{n\geq 0}\sup_{\lambda>0}\left[\lambda^p \mathbb{P}(|f_n| \geq \lambda)\right]^{1/p}$$

for the strong and the weak p-th norm of f. Following Burkholder (1984), we say that g is *differentially subordinate* to f, if for any $n \geq 0$ we have

$$|dg_n| \leq |df_n|$$

with probability 1. For example, this holds if g is a transform of f by a predictable sequence $v = (v_n)_{n\geq 0}$ taking values in $[-1, 1]$ (that is, we have $dg_n = v_n df_n$ for all n and by predictability of v we mean that each term v_n is measurable with respect to $\mathcal{F}_{(n-1)\vee 0}$). If $\mathbb{B}$ is a Hilbert space, then the differential subordination implies many interesting inequalities between f and g, which can be applied in many areas of mathematics. In addition, there is a beautiful method, due to Burkholder, which allows

E-mail address: ados@mimuw.edu.pl.
URL: http://www.mimuw.edu.pl/~ados.

0167-7152/$ – see front matter
doi:10.1016/j.spl.2011.12.007

to determine optimal constants in such estimates. The method rests on the existence of a certain special function, having appropriate convexity-type properties; for the detailed description and more on the subject, see the survey of Burkholder (1991). We will only mention here two types of inequalities, which are closely related to the results obtained in this paper. A celebrated L^p-inequality of Burkholder (1984) states that if $\mathbb{B}$ is a Hilbert space and g is differentially subordinate to f, then for any $1 < p < \infty$ we have the sharp bound

$$\|g\|_p \leq \max\left\{p-1, (p-1)^{-1}\right\}\|f\|_p. \tag{1}$$

For $p = 1$ the above inequality does not hold with any finite constant, but we have the weak-type bound $\|g\|_{1,\infty} \leq 2\|f\|_1$, and the constant 2 is optimal: see Burkholder (1984). In fact, Burkholder proved the sharp weak-type estimate for a wider range of parameters: if $1 \leq p \leq 2$, then for f, g as above,

$$\|g\|_{p,\infty} \leq \left(\frac{2}{\Gamma(p+1)}\right)^{1/p}\|f\|_p. \tag{2}$$

What are the best constants for $p > 2$? The answer is due to Suh (2005):

$$\|g\|_{p,\infty} \leq \left(p^{p-1}/2\right)^{1/p}\|f\|_p. \tag{3}$$

We continue this line of research and study the following novel estimates between the weak p-th norms of f and g. Until the end of this section, the filtration and the probability space are assumed to vary.

Theorem 1.1. *Suppose that $\mathbb{B}$ is a Hilbert space and f, g are $\mathbb{B}$-valued martingales such that g is differentially subordinate to f. Then for any $1 < p < \infty$ we have*

$$\|g\|_{p,\infty} \leq \frac{2p(p+1)}{p-1}\|f\|_{p,\infty}. \tag{4}$$

The constant is of optimal order $O(p)$ as $p \to \infty$ and $O((p-1)^{-1})$ as $p \to 1$, even in the special case when $\mathbb{B} = \mathbb{R}$ and g is a transform of f by a deterministic sequence with values in $[-1, 1]$.

Unfortunately, we have not managed to determine the sharp version of the estimate above. This is due to the fact that Burkholder's method, which is so efficient in the proofs of (1)–(3), does not seem to be applicable here (nevertheless, we use it to obtain some intermediate estimate; see Section 2).

One may wonder whether the above result can be carried over to a wider class of Banach spaces. Of course, if $\mathbb{B}$ is isomorphic to a Hilbert space, then the inequality (4) still holds true, possibly with a different constant. We will show that this implication can be reversed.

Theorem 1.2. *Suppose that a Banach space $\mathbb{B}$ has the following property: there are $p \in (1, \infty)$ and $C < \infty$ satisfying*

$$\|g\|_{p,\infty} \leq C\|f\|_{p,\infty} \tag{5}$$

for all $\mathbb{B}$-valued martingales f, g such that g is differentially subordinate to f. Then $\mathbb{B}$ is isomorphic to a Hilbert space.

However, if we restrict ourselves to the class of martingale transforms, we obtain a larger class of Banach spaces which are well-behaved with respect to (4). Recall that $\mathbb{B}$ is a UMD space if there is a constant $K = K(\mathbb{B})$ with the following property: if f is a $\mathbb{B}$-valued martingale and g is its transform by a real predictable sequence bounded in absolute value by 1, then

$$\lambda\mathbb{P}(g_n^* \geq \lambda) \leq K\|f_n\|_1$$

for every integer n and all $\lambda > 0$. For alternative definitions of UMD spaces and their geometrical characterizations, see e.g. Burkholder (1981) and Burkholder (2001).

Theorem 1.3. *The following conditions are equivalent.*

(i) *$\mathbb{B}$ is UMD.*

(ii) *There is a finite constant $\kappa(\mathbb{B})$ depending only on $\mathbb{B}$ such that*

$$\|g\|_{p,\infty} \leq \frac{\kappa(\mathbb{B})p^2}{p-1}\|f\|_{p,\infty}, \quad 1 < p < \infty, \tag{6}$$

whenever f is a $\mathcal{B}$-valued martingale and g is the transform of f by a predictable sequence with values in $[-1, 1]$.

(iii) *There are $p \in (1, \infty)$ and $\kappa < \infty$ such that*

$$\|g\|_{p,\infty} \leq \kappa\|f\|_{p,\infty} \tag{7}$$

whenever f is a $\mathcal{B}$-valued martingale and g is the transform of f by a deterministic sequence with values in $\{-1, 1\}$.

A few words about the organization of the paper. Theorem 1.1 is established in the next section, while the remaining results are proved in Section 3.