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Three random variable transformations giving Heisenberg-type uncertainty relations

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Abstract

Though intrinsically probabilistic, the 1927 quantum mechanics Heisenberg uncertainty relation is still non-existent as a native Kolmogorov probability construct. This paper fills the gap via three random variable transformations that generate Heisenberg-type uncertainty relations on Kolmogorov probability space (Ω, Σ, μ) without resorting to any quantum mechanics notion or formalism. A sufficient condition is given under which a class of Heisenberg uncertainty relations on (Ω, Σ, μ) is a bounded distributive lattice. Some applications in progress of the above results are anticipated.

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1. Introduction and summary

Though "intrinsically probabilistic" (Whittle, 1992, p. 283) the 1927 quantum mechanics Heisenberg uncertainty relation—HUR, for short—is still non-existent as a native Kolmogorov probability construct, i.e. as a construct not dependent on quantum mechanics itself (e.g., see

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Whittle, 1992, pp. 1–288). It is therefore noteworthy that—without resorting to any quantum mechanics notion or formalism—this paper:

(A) shows that more or less trivial HURs can be generated almost at will on Kolmogorov probability space (Ω, Σ, μ) (see Remark 1, Section 2), and moreover
(B) via three random variable (rv) transformations (see Theorems 1–3, Section 2), generates non-trivial HURs on (Ω, Σ, μ) that formally encompass the archetypical HUR of quantum mechanics (see Remark 2, Section 2).

The HUR on (Ω, Σ, μ) should be of interest to probabilists and statisticians also because of its striking formal resemblance to the Cramér–Rao inequality. Indeed, like the latter inequality, see Eq. (4.3.11) in Bickel and Doksum (1977, p. 129), the HUR too gives a lower bound to the product $\sigma_X \sigma_Y$ of the standard deviations of pairs (X, Y) of rv's in some given set. In fact (see Definition 1 Section 2) the HUR on (Ω, Σ, μ) is the inequality:

$$\sigma_X \sigma_Y \geqslant \alpha \quad \forall (X, Y) \in H_\alpha \subseteq S \times S = S^2, \ \ \alpha = \text{constant} \geqslant 0, \tag{1}$$

where $S \subset L^2(\Omega, \Sigma, \mu)$ is a set of non-degenerate square-integrable rv's on (Ω, Σ, μ), and $H_\alpha \subseteq S^2$ is the set of the pairs (X, Y) of rv's $X, Y \in S$ that satisfy inequality (1). For a quantum mechanics version of (1), see Eqs. (36)–(38) in Whittle (1992, p. 288), or Parthasarathy (1992, pp. 13–14).

Historically, the importance of HUR (1) arises from the applicative meaning of its consequence (3). Indeed, by (1), let

$$\sigma_X \sigma_Y = \sigma_{X'} \sigma_{Y'} = \gamma \geqslant \alpha \ (\sigma_X \neq \sigma_{X'}, \sigma_Y \neq \sigma_{Y'}, \gamma > 0) \quad (X, Y), (X', Y') \in H_\alpha. \tag{2}$$

Then, obviously, by (2), only one of the following two cases can ensue:

$$\begin{aligned} &\text{either} : \sigma_{X'} < \sigma_X \Leftrightarrow \sigma_{Y'} > \sigma_Y, \\ &\text{or} : \sigma_{X'} > \sigma_X \Leftrightarrow \sigma_{Y'} < \sigma_Y. \end{aligned} \tag{3}$$

The applicative meaning of (3) arises from the case of two random quantities (Q, P) all whose possible stochastic models are the pairs $(X, Y) \in H_\alpha$ such that $E(X) = q$ and $E(Y) = p \ \forall (X, Y) \in H_\alpha$, where q and p are two known or unknown fixed numbers. Then, (3) implies the puzzling (historically, epoch-making) fact that any increase in the accuracy about q (i.e. $\sigma_{X'} < \sigma_X$, see (3) first line) is necessarily associated with a decrease in the accuracy about p (i.e. $\sigma_{Y'} > \sigma_Y$, see (3) first line) and vice versa (see (3) second line)—whence, the terminology "uncertainty relation or principle" for inequality (1).

Theorems 4–6 in Section 2 deal with the set-theoretic and algebraic structure of the class of the HURs $H_\alpha \subseteq S^2$ on some (Ω, Σ, μ). In particular, Theorem 4 shows that such a class is closed w.r.t the set-theoretic converse of a set. Then, Theorem 5 specifies a sufficient condition under which the Cartesian product S^2 itself is a HUR. Finally, under the latter condition, Theorem 6 shows that the class of the HURs $H_\alpha \subseteq S^2$ is a bounded distributive lattice w.r.t set-theoretic union and intersection (it fails to be a Boolean algebra because of set-theoretic complementation). Some applications in progress of the above results are: (a) the assessment of the extent to which the Cramér–Rao inequality "is the HUR of mathematical statistics", (b) the extension of the bivariate HUR (1) to the n-dimensional case, (c) the introduction of HUR in economics, and (d) a contribution to the research on "Kolmogorov probability vs quantum probability".