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Comparison of level-crossing times for Markov and semi-Markov processes ☆

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Abstract

We derive sufficient conditions for the level-crossing ordering of continuous-time Markov chains (CTMCs) and semi-Markov processes (SMPs). The former ones constitute a relaxation of Kirstein's conditions for stochastic ordering of CTMCs in the usual sense, whereas the latter ones are an extension of conditions established by Di Crescenzo and Ricciardi and Irle for skip-free-to-the-right SMPs.

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1. Introduction

In the paper we address the level-crossing ordering, which compares stochastic processes in terms of their first passage times: a process X is said to be slower in level-crossing than Y if it takes X stochastically longer to exceed any given level than it does Y. To provide a clear definition of this ordering, we let Γ be either the set $\mathbb{N}$ of nonnegative integers or $\mathbb{R}_+ = [0, +\infty)$ and $S_y^W = \inf\{t \in \Gamma : W_t \geqslant y\}$ denote the hitting time of the set of values greater or equal to y by a stochastic process $W = (W_t)_{t\in\Gamma}$ with ordered state space. In addition, we let $S_{x,y}^W$ denote the hitting time of the set of values greater or equal to y when W departs from state x.

Definition 1. Let $X = (X_t)_{t\in\Gamma}$ and $Y = (Y_t)_{t\in\Gamma}$ be stochastic processes with common ordered state space I and $\leqslant_\star$ be a stochastic order relation for random variables. Then, X is said to be slower in level-crossing than Y in the $\star$-sense, denoted $X \leqslant_{\star\mathrm{lc}} Y$, if and only if $S_{x,y}^Y \leqslant_\star S_{x,y}^X$, for all $x, y \in I$.

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The level-crossing ordering was proposed by Irle and Gani (2001), for the case where random variables are compared through the usual (in distribution or strong) stochastic order, and was extended by Ferreira and Pacheco (2005a), for the case where random variables are compared through arbitrary stochastic orders. This ordering has been investigated by: Irle and Gani (2001), for skip-free to the right discrete-time Markov chains; Irle (2003), for skip-free to the right semi-Markov processes (SMPs) and continuous-time Markov chains (CTMCs), along with Brownian motions; Ferreira and Pacheco (2005b) for skip-free to the right CTMCs; and Ferreira and Pacheco (2005a) for DTMCs, SMPs and CTMCs such that the slowest of the compared processes is skip-free to the right.

In the paper we work with general (non-skip-free to the right) SMPs and CTMCs. In Section 2 we establish simple sufficient conditions for the level-crossing ordering of SMPs in IPICC (Integral stochastic orders for Positive random variables with Increasing functions Closed for Convolution) order senses, as defined below, via coupling [see, e.g., Lindvall, 2002; Thorisson, 2000]. In addition, the derived results are specialized to the case in which the compared SMPs are DTMCs, as well as the case in which they have the same embedded transition probability matrix, extending results of Di Crescenzo and Ricciardi (1996) and Irle (2003) for skip-free to the right SMPs.

Following the terminology due to Whitt (1986), an order $\leqslant_\star$ is an integral stochastic order if, given random variables W and Z, then $W \leqslant_\star Z$ if and only if $E[g(W)] \leqslant E[g(Z)]$ for all $g \in G_\star$, whenever the expected values exist, where $G_\star$ is some set of real functions. An order $\leqslant_\star$ is an IPICC order if it is an integral stochastic order such that $G_\star$ contains only functions that are increasing (more accurately, nondecreasing) on $\mathbb{R}_+$ and, moreover, $\leqslant_\star$ is closed for convolution. For the sake of completeness, we note that the class of IPICC orders includes, in particular, the usual order, as well as the Laplace transform, exponential, increasing convex, increasing concave, moments, expected value, and moment generating function orders (see, for example, Shaked and Shanthikumar, 1994; Müller and Stoyan, 2002 for details on the definitions and properties of these stochastic orders).

It is well known (see, e.g., Müller and Stoyan, 2002, Theorem 5.2.19 (a); Kulkarni, 1995, Theorem 6.29) that if X and Y are CTMCs with ordered state space I, then the two CTMCs are ordered in the usual sense, $X \leqslant_{\mathrm{st}} Y$, provided that their initial distributions are ordered in the same sense, $X_0 \leqslant_{\mathrm{st}} Y_0$, and their generator matrices, $\boldsymbol{Q}^X$ and $\boldsymbol{Q}^Y$, are ordered in the Kirstein's (Kirstein, 1976) sense, namely,

$$\sum_{m \geqslant n} q^X_{im} \leqslant \sum_{m \geqslant n} q^Y_{jm}, \quad i,j,n \in I \text{ such that } i \leqslant j \wedge (n \leqslant i \vee n > j). \tag{1}$$

In Section 3, the last section of the paper, we derive sufficient conditions for the level-crossing ordering of CTMCs in the usual sense that constitute a relaxation of Kirstein's conditions (1) for the stochastic ordering of CTMCs in the usual sense. Our derivation involves a coupling-adapted uniformization of the compared CTMCs and uses the sufficient conditions for the level-crossing ordering of SMPs in the usual sense derived in Section 2.

We end the Introduction with some notation used in the rest of the paper. Given a matrix $\boldsymbol{A} = (A_{ik})_{i,k \in I}$, we let $\boldsymbol{A}_{i\cdot} = (A_{ij})_{j \in I}$ denote the row vector containing row i of $\boldsymbol{A}$. Moreover, if $\boldsymbol{p}$ denotes a probability vector and Z denotes a random variable or distribution, like the exponential distribution with rate α ($\mathrm{Exp}(\alpha)$), then we let $\boldsymbol{p}^{-1}$ and Z^{-1} denote the generalized inverses of the distribution functions associated to $\boldsymbol{p}$ and Z, respectively. Furthermore, we assume in the rest of the paper that I is an ordered set, order-isomorphic to some bounded or unbounded interval of $\mathbb{Z}$, and let $\overline{I} = I \backslash \{\sup I\}$, where $\sup I$ is the supremum of the set I.

If $W = (W_t)_{t \in \Gamma}$ is a stochastic process with state space I, we let $W^{\leqslant y}$, $y \in I$, denote the process W restricted to the state space $\{x \in I : x \leqslant y\}$ in such a way that state y is made absorbing and all states of W greater or equal to y are collapsed into state y, namely: $W^{\leqslant y}_t = W_t$ if $t < S^W_y$, and $W^{\leqslant y}_t = y$ otherwise. Then, given a property $\triangle$, we say that W has the lower-$\triangle$ property if the process $W^{\leqslant y}$ has the $\triangle$ property, for all $y \in I$. Note that, in particular, a CTMC or an SMP may be lower-regular without being regular. Finally, we let $\geqslant_\star$ denote the reverse of $\leqslant_\star$ and $=_\star$ the stochastic equality in the $\star$-sense.

2. Semi-Markov processes

In this section we establish sufficient conditions for the level-crossing ordering of two SMPs in IPICC order senses, along with their translation for DTMCs and for SMPs with common embedded transition probability