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The overshoot of a random walk with negative drift

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Abstract

Let $\{S_n, n \geqslant 0\}$ be a random walk starting from 0 and drifting to $-\infty$, and let $\tau(x)$ be the first time when the random walk crosses a given level $x \geqslant 0$. Some asymptotics for the tail probability of the overshoot $S_{\tau(x)} - x$, associated with the event $(\tau(x) < \infty)$, are derived for the cases of heavy-tailed and light-tailed increments. In particular, the formulae obtained fulfill certain uniform requirements.

Keywords: Asymptotics; Ladder height; Tail probabilities; The class $\mathscr{S}(\gamma)$; Uniformity; Wiener–Hopf type factorization

1. Introduction

Let F be the common distribution function of increments of a random walk $\{S_n, n \geqslant 0\}$ starting at 0, with $\overline{F} = 1 - F$ satisfying $\overline{F}(x) > 0$ for all $x \in (-\infty, \infty)$. We assume that F has a finite mean $-\mu < 0$; hence, the random walk S_n drifts to $-\infty$ and its ultimate maximum

$$M = \max\{S_n, n \geqslant 0\}$$

is finite almost surely. Denote by

$$\tau(x) = \inf\{n \geqslant 1 : S_n > x\}, \quad x \geqslant 0,$$

the first time when the random walk $\{S_n, n \geqslant 0\}$ crosses a given level x, with the convention $\inf \phi = \infty$, and denote by

$$A(x) = S_{\tau(x)} - x$$

the overshoot of the random walk at the level x.

As remarked by Chang (1994), the overshoot is among the fundamental objects of study of random walk and renewal theory and therefore it plays an important role in a variety of fields of applied probability. In insurance, the quantity $A(x)$ may be interpreted as the deficit at ruin in the renewal model. In the present paper, we are interested in the tail probability of $A(x)$ associated with the event $(\tau(x) < \infty)$. Clearly, $\mathbb{P}(\tau(x) < \infty) > 0$ for all $x \geqslant 0$ since $\overline{F}(0) > 0$. We refer the reader to Janson (1986), Asmussen and Klüppelberg (1996), and Klüppelberg et al. (2004) for related discussions.

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Hereafter, for positive functions $a_i(\cdot)$ and $b_i(\cdot,\cdot)$, $i = 1, 2$, we write $a_1(x) \lesssim a_2(x)$ if $\limsup_{x\to\infty} a_1(x)/a_2(x) \leqslant 1$, write $a_1(x) \gtrsim a_2(x)$ if $\liminf_{x\to\infty} a_1(x)/a_2(x) \geqslant 1$, and write $a_1(x) \sim a_2(x)$ if both limits apply. Moreover, we say that $b_1(x,y) \sim b_2(x,y)$, as $x \to \infty$, holds uniformly for y in some nonempty set Δ if

$$\lim_{x\to\infty} \sup_{y\in\Delta} \left| \frac{b_1(x,y)}{b_2(x,y)} - 1 \right| = 0, \tag{1.1}$$

and we say that $b_1(x,y) \sim b_2(x,y)$, as $x \to \infty$, holds uniformly for $y \geqslant 0$ if relation (1.1) holds uniformly for $y \geqslant f(x)$ for any positive function $f(x) \to \infty$.

We shall apply the work of Veraverbeke (1977) to derive asymptotics for the tail probability $\mathbb{P}(A(x) > y, \tau(x) < \infty)$ under the assumption that the integrated tail distribution of F, defined by

$$F_I(x) = \frac{1}{\int_0^\infty \overline{F}(u)\,du} \int_0^x \overline{F}(u)\,du, \quad x \geqslant 0,$$

belongs to the class $\mathscr{S}(\gamma)$ for $\gamma \geqslant 0$; see below for its definition. The formula we obtain is

$$\mathbb{P}(A(x) > y, \tau(x) < \infty) \sim C \int_{x+y}^\infty \overline{F}(u)\,du \tag{1.2}$$

with $C > 0$ being explicitly expressed. We establish relation (1.2) in two limit senses: the one is $x \to \infty$ with requirement that relation (1.2) be uniform with respect to y in a relevant infinite interval, and the other is $y \to \infty$ with requirement that it be uniform with respect to x in a relevant infinite interval. As generally acknowledged, the uniformity often significantly merits the asymptotics obtained.

2. The main result

We say that a distribution F on $(-\infty, \infty)$ or its corresponding random variable X is defective (on the right) if $F(\infty) = \mathbb{P}(X < \infty) < 1$. In this case, its right tail is denoted by $\overline{F}(x) = F(\infty) - F(x)$. For two (possibly defective) distributions F, G, and a real number γ, denoted by $\widehat{F}(\gamma) = \int_{-\infty}^\infty e^{\gamma x} F(dx)$, if it exists, the moment generating function of F, by $F * G$ the convolution of F and G, and by F^{n*} the n-fold convolution of F for $n = 0, 1, 2, \ldots$, with F^{0*} taking unit mass at 0 and $F^{1*} = F$.

A distribution F on $(-\infty, \infty)$ is said to belong to the class $\mathscr{L}(\gamma)$, $\gamma \geqslant 0$, if

$$\lim_{x\to\infty} \frac{\overline{F}(x-u)}{\overline{F}(x)} = e^{\gamma u} \quad \text{for } u \in (-\infty, \infty). \tag{2.1}$$

Note that the convergence in (2.1) is automatically uniform on u in any finite interval. Furthermore, a distribution F on $[0, \infty)$ is said to belong to the class $\mathscr{S}(\gamma)$, $\gamma \geqslant 0$, if $F \in \mathscr{L}(\gamma)$ and

$$\lim_{x\to\infty} \frac{\overline{F^{2*}}(x)}{\overline{F}(x)} = 2\widehat{F}(\gamma) < \infty. \tag{2.2}$$

More generally, a (possibly defective) distribution F on $(-\infty, \infty)$ is still said to belong to the class $\mathscr{S}(\gamma)$, $\gamma \geqslant 0$, if the distribution $F^+(x) = F(x)/F(\infty)$, $x \geqslant 0$, belongs to this class. We remark that if $F \in \mathscr{S}(\gamma)$ then γ is the right abscissa of convergence of $\widehat{F}(\cdot)$. When $\gamma = 0$, relation (2.2) describes the famous subexponential class. Recent studies on these classes can be found in Rogozin (2000), Pakes (2004), Shimura and Watanabe (2005), and Tang (2006), among many others.

For a proper distribution F with finite mean $-\mu$, we make a convention that

$$\left. \frac{\gamma}{1 - \widehat{F}(\gamma)} \right|_{\gamma=0} = \frac{1}{\mu}. \tag{2.3}$$

Now we are ready to state the main result of this paper.