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Nonparametric estimation of volatility models with serially dependent innovations

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Abstract

We are interested in modelling the time series process $y_t = \sigma(x_t)\varepsilon_t$, where $\varepsilon_t = \phi_0\varepsilon_{t-1} + v_t$. This model is of interest as it provides a plausible linkage between risk and expected return of financial assets. Further, the model can serve as a vehicle for testing the martingale difference sequence hypothesis, which is typically uncritically adopted in financial time series models. When x_t has a fixed design, we provide a novel nonparametric estimator of the variance function based on the difference approach and establish its limiting properties. When x_t is strictly stationary on a strongly mixing base (hereby allowing for ARCH effects) the nonparametric variance function estimator by Fan and Yao [1998. Efficient estimation of conditional variance functions in stochastic regression. Biometrika 85, 645–660] can be applied and seems very promising. We propose a semiparametric estimator of ϕ_0 that is $\sqrt{T}$-consistent, adaptive, and asymptotic normally distributed under very general conditions on x_t.

Keywords: Weak form volatility models; Nonparametric/Semiparametric estimation; Asymptotics

1. Introduction

In this paper we consider estimation of a time series process with an unknown and possibly time varying conditional variance function and serially dependent innovations. By allowing for dependence in the innovation process, the model provides a plausible linkage between risk and expected return of financial assets not previously analyzed. Furthermore, the model provides a vehicle for testing the martingale difference sequence hypothesis, which is typically uncritically assumed in financial time series models, such as ARCH and GARCH.

We characterize the estimated parameters of the serially correlated innovation process as a solution to a weighted least squares (WLS) problem, where the weights are given by a nonparametric estimator of the conditional variance function. This semiparametric estimator belongs to the class of so-called MINPIN estimators. By using the framework of Andrews (1994) the asymptotic properties of the

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estimated parameters in the innovation process can be established under very general conditions. If the regressors entering the variance function are strictly stationary on an α-mixing base, the nonparametric estimator of the variance function suggested by Fan and Yao (1998) can be used. However, if the design is fixed, a new and in some cases more efficient nonparametric estimator is proposed and its asymptotic properties are established. Based on simulation experiments we show that under a fixed design this novel estimator has better small sample properties than the one proposed by Fan and Yao (1998).

2. The model

Consider the following process for the time series of interest denoted $y_t \in \mathbb{R}$, $t = 1, 2, \ldots, T$ where

$$y_t = \sigma_t \varepsilon_t, \tag{1}$$

$$\varepsilon_t = \phi_0 \varepsilon_{t-1} + v_t. \tag{2}$$

Furthermore, assume (i) $v_t \sim$i.i.d.$(0, 1)$, $E(|v_t|^{l+\gamma}) < \infty$ for $l = 1, \ldots, 4$ and for some $\gamma > 0$, (ii) $\phi_0 \in \Theta = (-1, 1)$, (iii) $\sigma_t^2 \equiv \sigma(x_t)^2 \in \mathscr{F} = C^2[0, 1]$, $P(\sigma_t^2 > 0) = 1$ for all $t = 1, 2, \ldots, T$, and finally (iv) ε_t is a strongly mixing sequence with mixing coefficient equal to $-(1 + 2/\delta)$ for $\delta > 0$. σ_t^2 (denoted also as σ^2) will be referred to as the variance function although strictly speaking it does not fully describe the variance structure of the model whenever $\phi_0 \neq 0$. It should be noticed that the model given by (1)–(2) belongs to the general class of function coefficient autoregressive (FAR) models, as can be seen from the following simple re-parameterization:

$$y_t = \sigma_t \sigma_{t-1}^{-1} \phi_0 y_{t-1} + \sigma_t v_t.$$

Here, the functional autoregressive coefficient is given by the term $g(x_t; \phi_0) = \sigma_t \sigma_{t-1}^{-1} \phi_0$. This coefficient is allowed to be numerically larger than unity for certain values of t, and during these periods y_t will exhibit explosive behavior. A second important feature of the model is that an increase in the variance will have a positive (negative) effect on the conditional expectation of y_t provided that ϕ_0 is positive (negative). If y_t are observations on a return series associated with a risky asset, this feature can be interpreted as a tradeoff between risk and expected return. The size and direction of such a tradeoff is of great importance in asset pricing theory and can easily be quantified using our approach. It is important to note that $\phi_0 \neq 0$ implies that the estimator of $\text{var}(y_t | x_t, x_{t-1}, y_{t-1})$ generally will be inconsistent, if based on residuals from a least squares regression of y_t on y_{t-1}, due to the time varying properties of the functional autoregressive coefficient. This potential source of inconsistency has often been ignored (e.g., when estimating (G)ARCH models), due to uncritical adoption of the assumption that the innovation process is a martingale difference sequence. As a by-product of our analysis, a simple parametric test of the martingale difference hypothesis, i.e., $\phi_0 = 0$, is proposed that enables the researcher to avoid this potential pitfall. Our main interest, however, is in the estimation of σ_t^2 and ϕ_0. We will proceed under the following two alternative assumptions regarding the regressor x_t:

Case 1: x_t has a fixed design on the unit interval.

Case 2: x_t is a strictly stationary process with an α-mixing base.

Note that Case 2 encompasses the situation where $x_t = y_{t-1}$, hence allowing for the presence of ARCH effects. The estimation procedure is simple and consists of two stages: In the first stage the estimator of σ_t^2—denoted $\widehat{\sigma}_t{}^2$—is obtained. Secondly, a semiparametric estimator of ϕ_0 is computed using WLS, where the weights are constructed using $\widehat{\sigma}_t{}^2$. This semiparametric estimator belongs to the class of MINPIN estimators introduced by Andrews (1994). In Case 1 we propose a novel nonparametric estimator based on the difference approach, which turns out to have nice asymptotic properties and is very easy to handle computationally. In Case 2 the estimator proposed by Fan and Yao (1998) seems promising. We begin, however, by characterizing the asymptotic properties of the MINPIN estimator.