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On the almost sure convergence for a linear process generated by negatively associated random variables in a Hilbert space

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Abstract

Let $\{\xi_k, k \in \mathbb{Z}\}$ be a strictly stationary negatively associated sequence of H-valued random variables with $E\xi_k = 0$, $E\|\xi_k\| < \infty$ and $E\|\xi_k\|^2 < \infty$ and $\{a_k, k \in \mathbb{Z}\}$ a sequence of linear operators such that $\sum_{j=0}^{\infty} j\|a_j\|_{L(H)} < \infty$. For a linear process $X_k = \sum_{j=0}^{\infty} a_j\xi_{k-j}$ we derive that $n^{-1}\sum_{k=1}^{n} X_k \to 0$ almost surely as $n \to \infty$.

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1. Introduction

Let H be a separable real Hilbert space with the norm $\|\cdot\|_H$ generated by an inner product, $\langle\cdot,\cdot\rangle_H$ and let $\{e_k, k \geq 1\}$ be an orthonormal basis in H. Let $L(H)$ be the class of bounded linear operators from H to H and denote by $\|\cdot\|_{L(H)}$ its usual norm. Let $\{\xi_k, k \in \mathbb{Z}\}$ be a strictly stationary sequence of H-valued random variables and $\{a_k, k \in \mathbb{Z}\}$ be a sequence of operators, $a_k \in L(H)$. We define the stationary Hilbert space process by:

$$X_k = \sum_{j=0}^{\infty} a_j\xi_{k-j}, \quad k \in \mathbb{Z}. \tag{1.1}$$

The sequence $\{X_k, k \in \mathbb{Z}\}$ is a natural extension of the multivariate linear processes (Brockwell and Davis, 1987, Chap. 11). We define

$$S_n = \sum_{k=1}^{n} X_k. \tag{1.2}$$

Notice that if $\sum_{j=0}^{\infty}\|a_j\|_{L(H)} < \infty$ and $\{\xi_k, k \in \mathbb{Z}\}$ is a sequence of H-valued i.i.d. random variables centered in $L_2(H)$, then it is well known that the series in (1.1) converges almost surely (Araujo and Gine, 1980, Chap. 3.2).

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Lehmann (1966) introduced the notion of positive(negative) quadrant dependence: Two random variables ξ_1 and ξ_2 are called positively(negatively) quadrant dependent if for all real α_1, α_2

$$P(\xi_1 > \alpha_1, \xi_2 > \alpha_2) \geq (\leq) P(\xi_1 > \alpha_1) P(\xi_2 > \alpha_2).$$

A finite family $\{\xi_i, 1 \leq i \leq n\}$ of real-valued random variables is said to be associated if for any coordinatewise increasing functions $f, g : \mathbb{R}^n \to \mathbb{R}$

$$\mathrm{Cov}(f(\xi_1, \ldots, \xi_n), g(\xi_1, \ldots, \xi_n)) \geq 0$$

whenever this covariance exists. A finite family $\{\xi_i, 1 \leq i \leq n\}$ is said to be negatively associated if for any disjoint subsets $A, B \subset \{1, \ldots, n\}$ and any real coordinatewise nondecreasing functions f on $\mathbb{R}^A$, g on $\mathbb{R}^B$,

$$\mathrm{Cov}(f(\xi_k, k \in A), g(\xi_k, k \in B)) \leq 0$$

whenever the covariance exists. An infinite family of random variables is associated(negatively associated) if every finite subfamily is associated(negatively associated). These concepts of dependence were introduced by Esary et al. (1967) and Joag-Dev and Proschan (1983), respectively.

Let us remark that associated(negatively associated) random variables are always pairwise positive(negative) quadrant dependent and that pairwise independent random variables are always pairwise positive quadrant dependent and associated(negatively associated).

Newman (1984) studied almost sure convergence for strictly stationary negatively associated sequence and Matula (1992) derived the strong law of large numbers for nonstationary negatively associated sequence.

As Burton et al. (1986) introduced the definition of association for random vectors we can give the definition of negative association for random vectors with values in $\mathbb{R}^d$.

Let $\{\xi_1, \ldots, \xi_m\}$ be a sequence of $\mathbb{R}^d$-valued random vectors. $\{\xi_1, \ldots, \xi_m\}$ is said to be associated if $\mathrm{Cov}(f(\xi_1, \ldots, \xi_m), g(\xi_1, \ldots, \xi_m)) \geq 0$ for any nondecreasing functions f and g on $\mathbb{R}^{md}$, such that the covariance exists and $\{\xi_1, \ldots, \xi_m\}$ is said to be negatively associated if $\mathrm{Cov}(f(\xi_k, k \in A), g(\xi_k, k \in B)) \leq 0$ for any disjoint subsets $A, B \subset \{1, \ldots, n\}$ and for any nondecreasing functions f on $\mathbb{R}^{d|A|}$, g on $\mathbb{R}^{d|B|}$ such that the covariance exists where $|A|$ is the cardinality of A.

We also extend the concept of association(negatively association) for random vectors with values in $\mathbb{R}^d$ to random vectors with values in a separable Hilbert space as follows : Let $\{\xi_n, n \geq 1\}$ be a sequence of random variables taking values in a Hilbert space $(H.\langle \cdot, \cdot \rangle)$. $\{\xi_n, n \geq 1\}$ is said to be associated(negatively associated) if for some orthonormal basis $\{e_k, k \geq 1\}$ of H and for any $d \geq 1$ the d-dimensional sequence $(\langle \xi_i, e_1 \rangle, \ldots, \langle \xi_i, e_d \rangle), i \geq 1$ is associated(negatively associated).

In Section 2 we will study the strong law of large numbers for negatively associated random variables in a Hilbert space and in Section 3 we derive the strong law of large numbers for a strictly stationary linear process generated by negatively associated random variables in a Hilbert space by applying this result.

2. Preliminaries

Lemma 2.1 (*Matula, 1992*). *Let $\{Y_n, n \geq 1\}$ be a sequence of negatively associated random variables with finite second moments and zero means. Then, for some $C > 0$,*

$$E\left(\max_{1 \leq k \leq n} \sum_{i=1}^{k} Y_i\right)^2 \leq C \sum_{i=1}^{n} E Y_i^2. \tag{2.1}$$

Lemma 2.2. *Let $\{\xi_n, n \geq 1\}$ be a negatively associated sequence of H-valued random variables with $E\xi_n = 0$ and $E\|\xi_n\|^2 < \infty, n \geq 1$. Then, for some $C > 0$*

$$E \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} \xi_i \right\|^2 \leq C \sum_{i=1}^{n} E\|\xi_i\|^2. \tag{2.2}$$