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ScienceDirect
Statistics & Probability Letters 78 (2008) 127–134
STATISTICS & PROBABILITY LETTERS
www.elsevier.com/locate/stapro

Improved minimax estimation of the bivariate normal precision matrix under the squared loss

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Received 18 October 2006; accepted 23 May 2007
Available online 8 June 2007

Abstract

Suppose that n independent observations are drawn from a multivariate normal distribution $\mathrm{N}_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$ with both mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ unknown. We consider the problem of estimating the precision matrix $\boldsymbol{\Sigma}^{-1}$ under the squared loss $L(\hat{\boldsymbol{\Sigma}}^{-1},\boldsymbol{\Sigma}^{-1}) = \mathrm{tr}(\hat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma} - \mathbf{I}_p)^2$. It is well known that the best lower triangular equivariant estimator of $\boldsymbol{\Sigma}^{-1}$ is minimax. In this paper, by using the information in the sample mean on $\boldsymbol{\Sigma}^{-1}$, we construct a new class of improved estimators over the best lower triangular equivariant minimax estimator of $\boldsymbol{\Sigma}^{-1}$ for $p = 2$. Our improved estimators are in the class of lower-triangular scale equivariant estimators and the method used is similar to that of Stein [1964. Inadmissibility of the usual estimator for the variance of a normal distribution with unknown mean. Ann. Inst. Statist. Math. 16, 155–160.]

MSC: 62F10; 62H12

Keywords: Precision matrix; Best lower triangular equivariant minimax estimator; Inadmissibility; Bivariate normal distribution; The squared loss

1. Introduction

Suppose that $\mathbf{X}_1,\ldots,\mathbf{X}_n$ are independent observations from a multivariate normal distribution $\mathrm{N}_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$, where both mean vector $\boldsymbol{\mu}\in\mathscr{R}^p$ and covariance matrix $\boldsymbol{\Sigma}>0$ are unknown. It is well known that $(\bar{\mathbf{X}},\mathbf{S})$ is a complete sufficient statistic for $(\boldsymbol{\mu},\boldsymbol{\Sigma})$, where

$$\bar{\mathbf{X}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i \quad \text{and} \quad \mathbf{S} = \sum_{i=1}^{n}(\mathbf{X}_i-\bar{\mathbf{X}})(\mathbf{X}_i-\bar{\mathbf{X}})^{\mathrm{T}}$$

denote the sample mean vector and the sample dispersion matrix, respectively, and T denotes the transpose of a matrix or vector. It is also well known that $\bar{\mathbf{X}}$ is independent of S and $\bar{\mathbf{X}}\sim\mathrm{N}_p(\boldsymbol{\mu},\boldsymbol{\Sigma}/n)$, $\mathbf{S}\sim W_p(n-1,\boldsymbol{\Sigma})$. Many

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0167-7152/$ - see front matter
doi:10.1016/j.spl.2007.05.010

authors have discussed the estimation of the generalized variance $|\mathbf{\Sigma}|$, the generalized precision $|\mathbf{\Sigma}|^{-1}$ and the covariance matrix $\mathbf{\Sigma}$, such as Kubokawa (1989), Kubokawa and Srivastava (2003), Pal (1988), Sinha and Ghosh (1987), Sugiura (1988), Sun (1998), Wang (1984), to mention just a few.

In this paper we consider the estimation of the precision matrix $\mathbf{\Sigma}^{-1}$ under the squared loss

$$L(\hat{\mathbf{\Sigma}}^{-1}, \mathbf{\Sigma}^{-1}) = \mathrm{tr}(\hat{\mathbf{\Sigma}}^{-1}\mathbf{\Sigma} - \mathbf{I}_p)^2. \tag{1}$$

As pointed out in Haff (1979), under the affine transformation group $\mathscr{A}$:

$$(\bar{\mathbf{X}}, \mathbf{S}) \to (\mathbf{A}\bar{\mathbf{X}} + \mathbf{b}, \mathbf{ASA}^{\mathrm{T}}); \quad (\boldsymbol{\mu}, \mathbf{\Sigma}) \to (\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A\Sigma A}^{\mathrm{T}}),$$

where $\mathbf{A}$ is an arbitrary $p \times p$ nonsingular matrix and $\mathbf{b}$ is an arbitrary $p \times 1$ vector, the best affine equivariant estimator (*BAEE*) is given by

$$\hat{\mathbf{\Sigma}}_1^{-1} = \frac{(n-p-4)(n-p-1)}{n-2}\mathbf{S}^{-1}. \tag{2}$$

Here we assume $n > p+4$ throughout this paper. Haff (1979) has also shown that the *BAEE* $\hat{\mathbf{\Sigma}}_1^{-1}$ is dominated by

$$\hat{\mathbf{\Sigma}}^{-1} = \frac{(n-p-4)(n-p-1)}{n-2}[\mathbf{S}^{-1} + vt(v)\mathbf{I}],$$

where $v = \{\mathrm{tr}(\mathbf{S})\}^{-1}$ and $t(v)$ is an absolutely continuous function of $v > 0$ such that $0 < t(v) \leqslant 2(p-1)/(n-p-1)$ and $t'(v) \leqslant 0$. Consider the lower triangular transformation group $\mathscr{B}$:

$$(\bar{\mathbf{X}}, \mathbf{S}) \to (\mathbf{G}\bar{\mathbf{X}} + \mathbf{h}, \mathbf{GSG}^{\mathrm{T}}); \quad (\boldsymbol{\mu}, \mathbf{\Sigma}) \to (\mathbf{G}\boldsymbol{\mu} + \mathbf{h}, \mathbf{G\Sigma G}^{\mathrm{T}}), \tag{3}$$

where $\mathbf{G}$ is a $p \times p$ lower-triangular matrix and $\mathbf{h}$ is a $p \times 1$ vector. It can be shown that the equivariant estimator under the above group has the form $\hat{\mathbf{\Sigma}}^{-1} = (\mathbf{K}^{-1})^{\mathrm{T}}\mathbf{\Delta}_p\mathbf{K}^{-1}$ where $\mathbf{K}$ is a lower triangular matrix with positive diagonal elements such that $\mathbf{KK}^{\mathrm{T}} = \mathbf{S}$ and $\mathbf{\Delta}_p = \mathrm{diag}(\delta_{1p}, \ldots, \delta_{pp})$ is an arbitrary diagonal matrix whose elements do not depend on $\mathbf{S}$. For $p = 2$, Olkin and Selliah (1977) and Sharma and Krishnamoorthy (1983) have derived the best lower triangular equivariant estimator (*BLEE*)

$$\hat{\mathbf{\Sigma}}_2^{-1} = (\mathbf{K}^{-1})^{\mathrm{T}}\mathbf{\Delta}_2\mathbf{K}^{-1}, \tag{4}$$

where $\mathbf{\Delta}_2 = \mathrm{diag}(\delta_{12}, \delta_{22})$ with δ_{12} and δ_{22} given by

$$\begin{cases} \delta_{12} = \dfrac{(n-2)(n-5)[(n-4)^2 - (n-6)]}{(n-2)(n-4)^2 - (n-6)}, \\ \delta_{22} = \dfrac{(n-3)(n-4)(n-5)(n-6)}{(n-2)(n-4)^2 - (n-6)}. \end{cases} \tag{5}$$

For general p, the *BLEE* of $\mathbf{\Sigma}^{-1}$ is much more complicated, one may refer to Krishnamoorthy and Gupta (1989). Note that the group $\mathscr{B}$ is a subgroup of the group $\mathscr{A}$, thus the *BLEE* $\hat{\mathbf{\Sigma}}_2^{-1}$ is an improved estimator of the *BAEE* $\hat{\mathbf{\Sigma}}_1^{-1}$. Also, since the lower triangular transformation group $\mathscr{B}$ is solvable, the *BLEE* $\hat{\mathbf{\Sigma}}_2^{-1}$ with constant risk is automatically minimax according to Kiefer (1957).

Using the approach in Gupta and Ofori-Nyarko (1995), we can get a better estimator than $\hat{\mathbf{\Sigma}}_2^{-1}$ in the form of the convex combination of the *BLEE* and the corresponding best upper triangular equivariant estimator (*BUEE*) of $\mathbf{\Sigma}^{-1}$. Moreover, Sharma and Krishnamoorthy (1983) constructed an orthogonal equivariant improved estimator of $\hat{\mathbf{\Sigma}}_2^{-1}$ for $p = 2$. For general $p > 2$, this becomes very complicated; one may refer to Takemura (1984). Here we note that all improved estimators of $\hat{\mathbf{\Sigma}}_2^{-1}$ given above depend on $\mathbf{S}$ only and the information in the sample mean $\bar{\mathbf{X}}$ on Σ is not utilized. It has been well known that when $\boldsymbol{\mu}$ and Σ are both unknown, the sample mean $\bar{\mathbf{X}}$ contains useful information about Σ. By using such information in $\bar{\mathbf{X}}$, Zhou et al. (2001) obtained an improved estimator that dominates the best lower triangular equivariant estimator of the precision matrix $\mathbf{\Sigma}^{-1}$ under the entropy loss. Kubokawa and Srivastava (2003) derived an estimator that dominates the best lower triangular equivariant estimator of the covariance matrix Σ under the entropy loss. It appears, however, that the problem of improving estimators of $\mathbf{\Sigma}^{-1}$ using the information in $\bar{\mathbf{X}}$ under the squared loss in (1) has not been considered in the literature. The main aim of this paper is to study this problem. In Section 2 below, under the squared loss L, by using Stein's method (see Stein, 1964), we will