Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Statistics & Probability Letters 76 (2006) 453–464
STATISTICS & PROBABILITY LETTERS
www.elsevier.com/locate/stapro

Stochastic structure of asymptotic quantization errors

Mykola Shykula[a], Oleg Seleznjev[a,b,*]

[a]*Department of Mathematics and Mathematical Statistics, Umeå University, SE-901 87 Umeå, Sweden*
[b]*Faculty of Mathematics and Mechanics, Moscow State University, 119 992 Moscow, Russia*

Received 28 October 2004
Available online 6 September 2005

Abstract

We consider quantization of continuous-valued random variables and processes in a probabilistic framework. Stochastic structure for non-uniform quantization errors is studied for a wide class of random variables. Asymptotic properties of the additive quantization noise model for a random process are derived for uniform and non-uniform quantizers. Some examples and numerical experiments demonstrating the rate of convergence in the obtained asymptotic results are presented.

MSC: primary 60G99; secondary 94A29; 94A34

Keywords: Distortion; Noise model; Optimal design; Scalar quantization; Random process

1. Introduction

Quantization of a continuous random variable X with known distribution density function $f(x)$ is an important problem which has been studied extensively last decades. Gray and Neuhoff (1998) provides a comprehensive overview of the problem. Later references can be found, for example, in Marco and Neuhoff (2004). For a given cellwidth $\varepsilon > 0$, an *uniform quantizer* $q_\varepsilon(X)$ and the quantization error $e_\varepsilon(X)$ are defined as

$$q_\varepsilon(X) := \varepsilon[X/\varepsilon] \quad \text{and} \quad e_\varepsilon(X) := X - q_\varepsilon(X) = \varepsilon\{X/\varepsilon\},$$

where $[x]$ and $\{x\}$ are the integer and fractional parts of x, respectively. For a wide class of density functions $f(x)$, the normalized error $z_\varepsilon(X) = e_\varepsilon(X)/\varepsilon = \{X/\varepsilon\}$ is known to be asymptotically uniform and independent of X as $\varepsilon \to 0$. Moreover, for certain random processes (or signals) $X(t), t \in [0, T]$, the normalized error process $Z_\varepsilon(t) := z_\varepsilon(X(t))$ behaves asymptotically like a white noise process, in particular, has asymptotically independent values for $t \neq s$, and so called additive *noise model* $X(t) = q_\varepsilon(X(t)) + \varepsilon Z_\varepsilon(t)$ holds.

*Corresponding address. Department of Mathematics and Mathematical Statistics, Umeå University, SE-901 87 Umeå, Sweden.
E-mail address: oleg.seleznjev@matstat.umu.se (O. Seleznjev).

0167-7152/$ - see front matter
doi:10.1016/j.spl.2005.08.022

For a random variable X with values in a finite interval, say, $[0,1]$, an n-level *non-uniform quantizer* is defined as

$$q_n(X) = q_n(X, D_n) = u_k \quad \text{if } u_k \leqslant X < u_{k+1}, \tag{1}$$

where $0 = u_0 < u_1 < \cdots < u_n = 1$ are the quantization points (or levels) and set up a design D_n.

The performance of the quantizer $q_n(X)$ is measured by the mean squared error (or *distortion*)

$$D(X, q_n(X)) := E(X - q_n(X))^2,$$

and similarly for $q_\varepsilon(X)$. For a fixed number of quantization points, the optimal designs have been found for some distributions (e.g., Gaussian). In the general case, the solution of the optimization problem is not straightforward. As in linear approximation with interpolation designs (see also Sacks and Ylvisaker, 1966; Su and Cambanis, 1993; Müller-Gronbach, 1996; Seleznjev, 2000; Hüsler et al., 2003) Cambanis and Gerr (1983) developed an approach with asymptotically optimal sequences of quantization designs. The sequence of quantization designs D_n^*, $n \geqslant 1$, is called asymptotically optimal for the class of designs $\mathscr{D}_n$ if

$$\lim_{n\to\infty} E(X - q_n(X, D_n^*))^2 \Big/ \inf_{D\in\mathscr{D}_n} E(X - q_n(X, D))^2 = 1.$$

Henceforth, for a bounded random variable $X \in [0,1]$, we consider the sequence of *regular* designs $D_n = D_n(h), n \geqslant 1$, generated by a positive continuous *quantization density function* $h(x), x \in [0,1]$,

$$\int_0^{u_k(n)} h(x)\,\mathrm{d}x = k/n, \quad k = 0, 1, \ldots, n. \tag{2}$$

In particular, for the uniform density function $h(x) \equiv 1, x \in [0,1]$, the quantization designs become equidistant (uniform) $u_{k+1}(n) - u_k(n) = 1/n, k = 0, \ldots, n-1$. We suppress the argument n for the design points $u_k = u_k(n)$, $k = 0, 1, \ldots, n$, from D_n, if doing so causes no confusion. Function $H(s) := \int_0^s h(x)\,\mathrm{d}x$, $s \in [0,1]$, provides a transformation from non-uniform to uniform quantization of a random variable with values in $[0,1]$ (see, e.g., Lee and Neuhoff, 1996). The asymptotically optimal quantization density function $h(x) = f(x)^{1/3} / \int_0^1 f(v)^{1/3}\,\mathrm{d}v$ in the class of regular n-points designs $\mathscr{D}_n$ in appropriate conditions is found in a number of papers (see, e.g., Linder, 1991, and the references therein). The stochastic structure of the asymptotic normalized quantization error $z_n(X) = n(X - q_n(X))$ and asymptotic properties of the corresponding additive noise non-uniform quantization model $X(t) = q_n(X(t)) + z_n(X(t))/n$ for a random process $X(t), t \in [0,T]$, are far less studied even for the Gaussian case. To the best of our knowledge, the only known result in this area is due to Lee and Neuhoff (1996) with rather approximate derivation of the distribution of the asymptotic quantization error. The discussion and some examples of quantization noise models are given in Kollár (1987). In our paper, we derive the stochastic structure of the asymptotic non-uniform quantization error and consider the quantization noise model for a wide class of random processes with rigorous arguments.

Of course, the study of uniform and non-uniform quantizers has been done, at least partly, in a number of papers (see Gray and Neuhoff, 1998, and references therein), but it seems worthwhile to gather some of these results and indicate some connections with other random process quantization studies. Moreover, the obtained results can be used for investigation of the non-uniform quantization of random process realizations. There are several approaches to these problems when the quantization distortion and the total number of quantization points (or *quantization rate*) are investigated. Fixed-rate quantizers use all codes (or codewords) of equal length, whereas in variable-rate quantization, the rate is a random variable. Some problems for the fixed-rate quantization of Gaussian processes and optimal properties of the corresponding Karhunen–Loève expansion are considered in Luschgy and Pagès (2002). In the conventional digital quantization setting, first a signal is discretized in time and then in amplitude for a given time usually ignoring the stochastic structure of the original random process. The different variable-rate approach is developed in Shykula and Seleznjev (2004) for the uniform quantization of realizations of Gaussian processes.

Let $\xrightarrow{\mathrm{d}}$ denote convergence in distribution. Denote by $F_X(x)$, $F_Y(x)$, and $F_{X,Y}(x,y)$ the distribution functions of random variables X, Y, and their joint distribution function, respectively. For a set of random variables X_ε, Y_ε, $\varepsilon > 0$, let $X_\varepsilon \xrightarrow{\mathrm{d}} X$, $Y_\varepsilon \xrightarrow{\mathrm{d}} Y$, and $(X_\varepsilon, Y_\varepsilon) \xrightarrow{\mathrm{d}} (X, Y)$ as $\varepsilon \to 0$. Then we say that X_ε is