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A converse to precise asymptotic results

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Abstract

Let $\{X, X_n, n \geqslant 1\}$ be i.i.d. random variables with partial sums $\{S_n, n \geqslant 1\}$, put $f(\varepsilon) = \sum_n a_n P(|S_n| \geqslant \varepsilon b_n)$, $\varepsilon \geqslant 0$, and assume there exist functions g and h, such that $\lim_{\varepsilon \searrow 0} g(\varepsilon)f(\varepsilon) = h(EX^2)$ whenever $EX^2 < \infty$ and $EX = 0$. We prove the converse result, namely that $\limsup_{\varepsilon \searrow 0} g(\varepsilon)f(\varepsilon) < \infty$ and $b_n = O(n)$ imply $EX^2 < \infty$ and $EX = 0$.

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1. Introduction and result

Let $\{X, X_n, n \geqslant 1\}$ be i.i.d. random variables with $P(X \neq 0) > 0$ and partial sums $\{S_n, n \geqslant 1\}$, and consider series of the type

$$f(\varepsilon) = \sum_n a_n P(|S_n| \geqslant \varepsilon b_n), \quad \varepsilon > 0,$$

where $a_n, b_n > 0$ and $\sum_n a_n = \infty$. Then there exists a threshold α, such that $f(\varepsilon) = \infty$ for $\varepsilon < \alpha$, while $f(\varepsilon) < \infty$ for $\varepsilon > \alpha$. The so-called precise asymptotic problem consists in finding, under appropriate moment conditions, an elementary function $g(\varepsilon) > 0, \varepsilon > \alpha$, such that $\lim_{\varepsilon \searrow \alpha} g(\varepsilon) = 0$ and $\lim_{\varepsilon \searrow \alpha} g(\varepsilon)f(\varepsilon) = l \neq 0, \infty$, i.e., in establishing that $f(\varepsilon) \sim l/g(\varepsilon)$ as $\varepsilon \searrow \alpha$. For almost exhaustive references on this area, see Spătaru (2004a,b). Except for the cases, where X is assumed to belong to the domain of attraction of a stable law, most known such asymptotics are derived by assuming at least $EX^2 < \infty$, and then it turns out that $l = h(EX^2)$ for some finite function h. Typical precise asymptotic results suited to have genuine converses are those where the assumption "$EX^2 < \infty$ and $EX = 0$" is stronger than "$f(\varepsilon) < \infty$ for some $\varepsilon > 0$". When $\alpha = 0$, such a general result is as follows.

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Theorem A. *There exist positive and finite functions g and h on* $(0,\infty)$ *with* $\lim_{x\to\infty} h(x)=\infty$, *depending solely on the sequences* $\{a_n\}$ *and* $\{b_n\}$, *such that*

$$\lim_{\varepsilon\searrow 0} g(\varepsilon)f(\varepsilon)=h(EX^2),$$

whenever $EX^2<\infty$ *and* $EX=0$.

We obtain the next converse to Theorem A.

Theorem 1. *Let g be as in Theorem* A, *and assume that*

$$\limsup_{\varepsilon\searrow 0} g(\varepsilon)\sum_n a_nP(|S_n|\geqslant \varepsilon b_n)<\infty, \tag{1}$$

where $b_n=\mathrm{O}(n)$. *Then* $EX^2<\infty$ *and* $EX=0$.

Remark 1. (1) implies that $f(\varepsilon)<\infty$ for all $\varepsilon>0$. If this, in turn, implied that $EX^2<\infty$ and $EX=0$, there would be nothing more to prove. The two examples below are veritable converses to important precise asymptotic theorems.

Proof. We first show that (1) implies $EX^2<\infty$. To do this, let $\{X', X'_n, n\geqslant 1\}$ be an independent copy of $\{X, X_n, n\geqslant 1\}$, and consider the symmetrized random variables $Y=X-X'$, $Y_n=X_n-X'_n$, $U_n=Y_1+\cdots+Y_n$, $n\geqslant 1$. Next, for any $\lambda>0$, write $Y_n(\lambda)=Y_nI\{|Y_n|\leqslant\lambda\}-Y_nI\{|Y_n|>\lambda\}$, $n\geqslant 1$. Since each Y_n is symmetric, it follows that $\{Y, Y_n(\lambda), n\geqslant 1\}$ are i.i.d. random variables. Clearly, (1) implies that

$$\limsup_{\varepsilon\searrow 0} g(\varepsilon)\sum_n a_nP(|U_n|\geqslant 2\varepsilon b_n)<\infty. \tag{2}$$

Since

$$\sum_{k=1}^{n} Y_kI\{|Y_k|\leqslant\lambda\}=\left(U_n+\sum_{k=1}^{n} Y_k(\lambda)\right)\Big/2,$$

(2) shows that

$$\begin{aligned}&\limsup_{\varepsilon\searrow 0} g(\varepsilon)\sum_n a_nP\left(\left|\sum_{k=1}^{n} Y_kI\{|Y_k|\leqslant\lambda\}\right|\geqslant 2\varepsilon b_n\right)\\ &\quad\leqslant 2\limsup_{\varepsilon\searrow 0} g(\varepsilon)\sum_n a_nP(|U_n|\geqslant 2\varepsilon b_n)<\infty.\end{aligned} \tag{3}$$

On the other hand, by Theorem A, we have

$$\lim_{\varepsilon\searrow 0} g(\varepsilon)\sum_n a_nP\left(\left|\sum_{k=1}^{n} Y_kI\{|Y_k|\leqslant\lambda\}\right|\geqslant 2\varepsilon b_n\right)=h(E[Y^2I\{|Y|\leqslant\lambda\}/4]). \tag{4}$$

Combining (3) with (4), it follows there exists $M>0$ such that

$$h(E[Y^2I\{|Y|\leqslant\lambda\}]/4)\leqslant M<\infty,\quad \lambda>0. \tag{5}$$

As $\lim_{x\to\infty} h(x)=\infty$, on letting $\lambda\to\infty$ in (5) shows that $EY^2<\infty$, and so $EX^2<\infty$.

We now show that (1) implies $EX=0$. Note that $nEX=S_n-(S_n-nEX)$, $n\geqslant 1$. Hence, by (1) and Theorem A, we have

$$\limsup_{\varepsilon\searrow 0} g(\varepsilon)\sum_n a_nP(|nEX|\geqslant 2\varepsilon b_n)<\infty. \tag{6}$$

Since $\sum_n a_n=\infty$ and $b_n=\mathrm{O}(n)$, (6) leads to $EX=0$. $\square$